

# Cateogry

Shinji KONO <kono@ie.u-ryukyu.ac.jp>

e-mail:kono@ie.u-ryukyu.ac.jp

Information Engineering, University of the Ryukyus  
Nishihara-cyo 1, Okinawa, 903-01, Japan

August 24, 2013

## 1 Adjunction

$$U\varepsilon \circ \eta U = 1_U$$

$$\varepsilon F \circ F\eta = 1_F$$

$$f : a \rightarrow Ub$$

$$\begin{array}{ccc} & & FU(b) \\ & \nearrow^{F(f)} & \downarrow \varepsilon(b) \\ F(a) & \xrightarrow{f^*} & b \end{array}$$

$$\begin{array}{ccc} UF(a) & \xrightarrow{U(f^*)} & Ub \\ \eta(a) \uparrow & \nearrow f & \\ a & & \end{array}$$

Universal mapping problem is for each  $f : a \rightarrow Ub$ , there exists  $f^*$  such that  $f = U(f^*)\eta$ .

## 2 Adjunction to Universal mapping

In adjunction  $(F, U, \varepsilon, \eta)$ , put  $f^* = \varepsilon(b)F(f)$ , we are going to prove  $f^*$  is a solution of Universal mapping problem. That is  $U(f^*)\eta = f$ .

$$\begin{array}{ccc}
 & \xrightarrow{U(\varepsilon(b)F(f))=U(f^*)} & \\
 UF(a) & \xrightarrow{UF(f)} UFUb & \xrightarrow{U(\varepsilon(b))} ? \\
 \eta(a) \uparrow & & \uparrow \eta(Ub) \\
 a & \xrightarrow{f} & Ub
 \end{array}$$

$$\begin{array}{ccc}
 UF(a) & \xrightarrow{UF(f)} & UFUb \\
 \eta(a) \uparrow & \searrow U(f^*) & \uparrow \eta(Ub) \\
 a & \xrightarrow{f} & Ub
 \end{array}
 \quad \left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) U(\varepsilon(b))$$

$$\begin{array}{ccc}
 F(a) & \xrightarrow{F(f)} & FU(b) \\
 & \searrow f^* & \downarrow \varepsilon(b) \\
 & & b
 \end{array}
 \quad U\varepsilon \circ \eta U = 1_U$$

$$U(\varepsilon(b))\eta(U(b)) = 1_{U(b)}$$

means that

$\varepsilon(b) : FU(b) \rightarrow b$  is solution of  $1_{U(b)}$ .

naturality  $f\eta(U(b)) = U(F(f))\eta(a)$

gives solution  $U(\varepsilon(b))UF(f) = U(F(f))\varepsilon(b)$  for  $f$ .

$$U(f^*)\eta(a)(a) = f(a)$$

then

$$U(\varepsilon(b)F(f))\eta(a)(a) = U(\varepsilon(b))UF(f)\eta(a)(a)$$

since  $F$  is a functor. And we have

$$U(\varepsilon(b))UF(f)\eta(a)(a) = U(\varepsilon(b))\eta(b)f(a)$$

because of naturality of  $\eta$

$$\begin{array}{ccc} UF(a) & \xleftarrow{\eta(a)} & a & & UF(f)\eta(a) = \eta(b)f \\ & \downarrow UF(f) & \downarrow f & & \\ UF(b) & \xleftarrow{\eta(b)} & b & & \end{array}$$

too bad.... we need some thing more.

## 2.1 Adjoint of

$$U\varepsilon \circ \eta U = 1_U$$

$$F(a) \xrightarrow{F(f)} FU(b) \xrightarrow{\varepsilon(b)} b$$

$$UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

$$a \xrightarrow{\eta(a)} UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

$$a \xrightarrow{f} Ub \xrightarrow[\eta(Ub)]{\eta(Ub)} UFU(b) \xrightarrow[U(\varepsilon(b)=1)]{U(\varepsilon(b))} U(b)$$

$$U\varepsilon \circ \eta U = 1_U$$

naturality of  $f : a \rightarrow Ub$

$$\begin{array}{ccc} Ub & \xrightarrow{\eta(Ub)} & UF(Ub) \\ \uparrow f & & \uparrow UF(f) \\ a & \xrightarrow{\eta(a)} & UF(a) \end{array}$$

$$\begin{array}{ccc}
UF(a) & \xrightarrow{UF(f)} & UF(U(b)) \\
\uparrow \eta(a) & & \uparrow \eta(U(b)) \\
a & \xrightarrow{f} & U(b)
\end{array}
\quad
\begin{array}{c}
UF(U(b)) \\
\downarrow U(\varepsilon(U(b))) \\
U(b)
\end{array}$$

Solution of universal mapping yields naturality of  $U\varepsilon \circ \eta U = 1_U$ .  
 $\varepsilon F \circ F\eta = 1_F$ .

$$\begin{array}{ccccc}
UF(a) & & F(a) & \xrightarrow{F(\eta(a))} & FUF(a) \\
\uparrow \eta(a) & \searrow 1_{UF(a)} & & \searrow 1_{F(a)} & \downarrow \varepsilon(F(a)) \\
a & \xrightarrow{\eta(a)} & UF(a) & \xrightarrow{(1_{UF(a)})^*} & F(a)
\end{array}$$

### 3 Universal mapping to adjunction

Functor  $U$ , mapping  $F(a)$  and  $(f)^*$ ,  $U(f^*)\eta(a) = f$  are given.  
 object  $F(a) : A \rightarrow B$   
 $\eta(a) : a \rightarrow UF(a)$   
 put

$$\begin{aligned}
F(f) &= (\eta(b)f)^* \\
\varepsilon &: FU \rightarrow 1_B \\
\varepsilon(b) &= (1_{U(b)})^*
\end{aligned}$$

$$F(a) \xrightarrow{f^*} b$$

$$\begin{array}{ccc}
UF(a) & \xrightarrow{U(f^*)} & Ub \\
\uparrow \eta(a) & \nearrow f & \\
a & & 
\end{array}$$

$$f = U(f^*)\eta$$

Show  $F$  is a Functor, that is  $F(fg) = F(f)F(g)$ .

Show naturality of  $\eta(a)$ .

$$f : a \rightarrow b, F(f) = (\eta(b)f)*$$

Show naturality of  $\varepsilon(b) = (1_U)*$ .

### 3.1 Definitions

f's destination

$$f : a \rightarrow U(b)$$

universal mapping

$$U(f*)\eta(a) = f$$

definition of F(f)

$$F(f) = (\eta(U(b))f)*$$

definition of  $\varepsilon$

$$\varepsilon(b) = (1_{U(b)})*$$

$$\begin{array}{ccc}
 FUF(a) & \xrightarrow{FU(f*)} & FU(b) \\
 F(\eta(a)) \left( \begin{array}{c} \uparrow \\ \varepsilon(F(a)) \\ \downarrow \end{array} \right) & \nearrow F(f) & \downarrow \varepsilon(b)=(1_U)* \\
 F(a) & \xrightarrow{f*} & b
 \end{array}$$

$$\begin{array}{ccc}
 UF(a) & \xrightarrow{UF(f)} & UFU(b) \\
 \eta(a) \left( \begin{array}{c} \uparrow \\ U(f*) \\ \downarrow \end{array} \right) & \searrow \eta(U(b)) & \left( \begin{array}{c} \uparrow \\ U(\varepsilon(b)) \\ \downarrow \end{array} \right) \\
 a & \xrightarrow{f} & U(b)
 \end{array}$$

$$\begin{aligned}
 \varepsilon F \circ F \eta &= 1_F, \varepsilon(b) = (1_{U(b)})*, \\
 \varepsilon(F(a)) &= (1_{UF(a)})*
 \end{aligned}$$

$$\begin{array}{ccccc}
UF(a) & & F(a) & \xrightarrow{F(\eta(a))} & FUF(a) \\
\eta(a) \uparrow & \searrow^{U(1_{F(a)})} & & \searrow^{1_{F(a)}} & \downarrow^{\varepsilon(F(a))} \\
a & \xrightarrow{\eta(a)} & U(F(a)) & & F(a)
\end{array}$$

### 3.2 Functor $F$

$$F(f) = (\eta(b)f)^*$$

$$U(F(f))\eta(a) = \eta(b)f$$

show  $F(fg) = F(f)F(g)$

$$a \xrightarrow{g} Ub \xrightarrow{f} U Uc$$

$$\begin{aligned}
U(F(g))\eta(a) &= \eta(Ub)g \\
U(F(f))\eta(Ub) &= \eta(UUc)f
\end{aligned}$$

show

$$U(F(f)F(g))\eta(a) = \eta(UUc)fg$$

then  $F(f)F(g) = F(fg)$

$$\begin{aligned}
U(F(f)F(g))\eta(a) &= UF(f)UF(g)\eta(a) \\
&= UF(f)\eta(Ub)g \\
&= \eta(UUc)fg
\end{aligned}$$

Q.E.D.

$$\begin{array}{ccccc}
F(a) & \xrightarrow{F(g)} & FU(b) & \xrightarrow{FU(f)} & FUU(c) \\
& \searrow^{g*} & \downarrow \varepsilon(b) & \searrow^{f*} & \downarrow \varepsilon(Uc) \\
& & b & & U(c)
\end{array}$$

$$\begin{array}{ccccc}
UF(a) & \xrightarrow{UFg} & UFUb & \xrightarrow{UFf} & UFUUC \\
\eta(a) \uparrow & \searrow^{U(g*)} & \uparrow \eta(Ub) & \searrow^{Uf*} & \uparrow \eta(UUc) \\
a & \xrightarrow{g} & Ub & \xrightarrow{f} & UU(c)
\end{array}$$

### 3.3 naturality of

$$\eta : 1 \rightarrow UB$$

$$\begin{array}{ccc}
UF(a) & \xrightarrow{UF(f)} & UFb \\
\eta(a) \uparrow & & \uparrow \eta(b) \\
a & \xrightarrow{f} & b
\end{array}$$

$$\text{prove } \eta(b)f = UF(f)\eta(a)$$

$$\begin{aligned}
\eta(b)f & : a \rightarrow UFb \\
F(f) & = (\eta(b)f) * \quad (\text{definition}) \\
\eta(b)f & = U(F(f))\eta(a)
\end{aligned}$$

Q.E.D.

### 3.4 naturality of

$$\varepsilon : FU \rightarrow 1_B$$

$$U : B \rightarrow A$$

$$\begin{aligned}
\varepsilon(b) & = (1_{U(b)}) * \\
U(\varepsilon(b))\eta(U(b)) & = 1_{U(b)}
\end{aligned}$$

$$U(\varepsilon(b))\eta(U(b))U(b) = U(b)$$

$$\begin{array}{ccc} FU(b) & \xrightarrow{FU(f)} & FU(c) \\ \downarrow \varepsilon(b) & & \downarrow \varepsilon(c) \\ b & \xrightarrow{f} & c \end{array}$$

prove  $f\varepsilon(b) = \varepsilon(c)FU(f)$

$$f = Ub \rightarrow Uc$$

$$F(Ub) \xrightarrow{(1_{Ub})^*} b \xrightarrow{f} c$$

$$\begin{array}{ccccc} UF(Ub) & \xrightarrow{U(1_{Ub})^*} & Ub & \xrightarrow{U(f)} & U(c) \\ \uparrow \eta(Ub) & \nearrow UFU(f) & \nearrow U(1_{U(c)^*}) & \nearrow & \nearrow \\ & & UFUc & & \\ & & \uparrow \eta(Uc) & & \\ Ub & \xrightarrow{U(f)} & Uc & & \end{array}$$

$$F(Ub) \xrightarrow{FU(f)} FU(c) \xrightarrow{(1_{U(c)})^*} c$$

show  $\varepsilon(c)FU(f)$  and  $f\varepsilon(b)$  are both solution of  $(1_{Uc})U(f)(= U(f)(1_{Ub}))$

$$\begin{aligned} (f\varepsilon(b))\eta(Ub)Ub &= U(f)U(\varepsilon(b))\eta(Ub)Ub \\ &= U(f)1_{Ub}Ub = U(f)Ub = Ufb = U(f)(1_{Ub})Ub \\ f\varepsilon(b) &= (U(f)(1_{Ub}))^* \end{aligned}$$

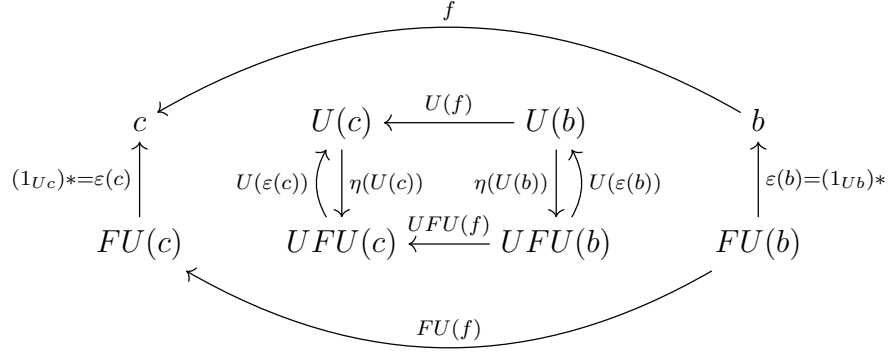
$UFU(f)\eta(Ub) = \eta(Uc)U(f)$  naturality of

$$\begin{aligned} U(\varepsilon(c)FU(f))\eta(Ub)Ub &= U(\varepsilon(c))UFU(f)\eta(Ub)Ub \\ &= U(\varepsilon(c))\eta(Uc)U(f)Ub = 1_{U(c)}U(f)Ub = U(f)Ub = U(f)(1_{Ub})Ub \end{aligned}$$



$$U(\varepsilon(c))\eta(Uc) = 1_U(c)$$

end of proof.



It also prove

$$U\varepsilon \circ \eta U = 1_U$$

### 3.5 $U\varepsilon \circ \eta U = 1_U$

$$\varepsilon(b) = (1_U(b))^*$$

that is

$$U((1_U(b))^*\eta(U(b))) = 1_U(b) \quad U(\varepsilon(b))\eta(U(b)) = 1_U(b)$$

$$U\varepsilon \circ \eta U = 1_U$$

### 3.6 $\varepsilon F \circ F\eta = 1_F$

$$\eta(a) = U(1_F(a))\eta(a)$$

$$\Rightarrow (\eta(a))^* = 1_F(a) \dots (1)$$

$$\varepsilon(F(a)) = (1_U F(a))^*$$

$$\Rightarrow 1_U F(a) = U(\varepsilon(F(a)))\eta(UF(a))$$

times  $\eta(a)$  from left

$$\eta(a) = U(\varepsilon(F(a)))\eta(UF(a))\eta(a)$$

$$\eta(UF(a)) = UF\eta(a) \text{ naturality of } \eta$$

$$\eta(a) = U(\varepsilon(F(a)))(UF\eta(a))\eta(a)$$

$$= U(\varepsilon(F(a))F\eta(a))\eta(a)$$

$$\Rightarrow (\eta(a))^* = \varepsilon(F(a))F\eta(a) \dots (2)$$

from (1),(2), since  $(\eta(a))^*$  is unique

$$\varepsilon(F(a))F\eta(a) = 1_{F(a)}$$

$$\begin{array}{ccccc}
 UF(a) & \xrightarrow{F} & FUF(a) & \xrightarrow{U} & UFUF(a) \\
 \eta(a) \uparrow & & \begin{array}{c} \uparrow \\ F(\eta(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ \varepsilon(F(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ \eta(UF(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ U(\varepsilon F(a)) \\ \downarrow \end{array} \\
 a & \xrightarrow{F} & F(a) & \xrightarrow{U} & UF(a) \\
 & & (\eta(a))* = 1_{Fa} \uparrow & & \uparrow \eta(a) = U(\varepsilon F(a))\eta(UF(a)) \\
 & & F(a) & \xrightarrow{U} & UF(a)
 \end{array}$$

$$\begin{array}{ccc}
 UF(a) & \xrightarrow{F} & FUF(a) & \xrightarrow{U} & UFUF(a) & & FUF(a) \\
 \uparrow \eta(a) & & \begin{array}{c} \uparrow \\ F(\eta(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ \varepsilon(F(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ U(\varepsilon F(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ \eta(UF(a)) \\ \downarrow \end{array} & \begin{array}{c} \uparrow \\ \varepsilon(F(a)) \\ \downarrow \end{array} \\
 a & \xrightarrow{F} & F(a) & \xrightarrow{U} & UF(a) & & F(a)
 \end{array}$$

$$\begin{array}{c}
 UUF(a) \\
 \eta(UF(a)) \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ U(\varepsilon(Fa)) \\ \downarrow \end{array} U(\varepsilon(F(a)))\eta(UF(a)) = 1_U F(a) \\
 UF(a)
 \end{array}$$

$$\varepsilon(F(a)) = (1_U F(a))*$$

$$\begin{array}{ccc}
 FA & \longrightarrow & UFA \\
 \downarrow F\eta(A) & & \downarrow U\eta A \\
 FUF(A) & \longrightarrow & UFUFA
 \end{array}$$

$\varepsilon(F(A))$  の定義から  $U(\varepsilon(F(A))) : UFUFA \rightarrow UFA$   
 唯一性から  $\varepsilon(F(A)) : FUF(A) \rightarrow FA$  従って

$$\varepsilon(F(A))F\eta(A) = 1$$

ってなのを考えました。

$$\begin{array}{l}
 U\eta(A') = U(1_{FA'})\eta(A') \text{ より} \\
 \eta(A')* = 1_{FA'},
 \end{array}$$

$U\eta(A') = U(\varepsilon(F A')F\eta(A'))\eta(A')$  より  
 $\eta(A')^* = \varepsilon(F A')F\eta(A')$  から  
 $1_F = \varepsilon F.F\eta$  は言えました。

後方で  $\eta$  の自然性と  $\varepsilon$  の定義を使いました。

## 4 おまけ

$$\varepsilon F \circ F\eta = 1_F, U\varepsilon \circ \eta U = 1_U$$

$$\begin{array}{ccc}
 UFU(a) & \xleftarrow{U} & FU(a) \\
 \eta(U(a)) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \downarrow \varepsilon(a) \\
 U(a) & \xleftarrow{U} & (a)
 \end{array}$$

$$\begin{array}{ccc}
 FUF(a) & \xleftarrow{F} & UF(a) \\
 F\eta(a) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \uparrow \eta(a) \\
 F(a) & \xleftarrow{F} & (a)
 \end{array}$$

なら、  $FU(\varepsilon(F(a))) = \varepsilon F(a)$  ?

$$\begin{array}{ccc}
UFU(F(a)) & \xleftarrow{U} & FU(F(a)) \\
\eta(U(a)) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \downarrow \varepsilon(F(a)) \\
U(F(a)) & \xleftarrow{U} & F(a)
\end{array}$$

$$\begin{array}{ccc}
FUFU(F(a)) & \xleftarrow{FU} & FU(F(a)) \\
F\eta(U(a)) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \downarrow \varepsilon(F(a)) \\
FU(F(a)) & \xleftarrow{FU} & F(a)
\end{array}$$

$$\begin{array}{ccc}
FUF(a) & \xleftarrow{F} & UF(a) \\
F\eta(a) \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \uparrow \eta(a) \\
F(a) & \xleftarrow{F} & (a)
\end{array}$$

## 5 Monad

$(T, \eta, \mu)$

$$T : A \rightarrow A$$

$$\eta : 1_A \rightarrow T$$

$$\mu : T^2 \rightarrow T$$

$$\mu \circ T\eta = 1_T = \mu \circ \eta T \text{ Unity law}$$

$$\mu \circ \mu T = \mu \circ T\mu \text{ association law}$$

$$\begin{array}{ccc}
T & \xrightarrow{T\eta} & T^2 & & T^3 & \xrightarrow{\mu T} & T^2 \\
\eta T \downarrow & \searrow 1_T & \downarrow \mu & & T\mu \downarrow & & \downarrow \mu \\
T^2 & \xrightarrow{\mu} & T & & T^2 & \xrightarrow{\mu} & T
\end{array}$$

## 6 Adjoint to Monad

Monad  $(UF, \mu, U \dashv F)$  on adjoint  $(U, F, \eta, \varepsilon)$

$$\varepsilon F \circ F \mu = 1_F$$

$$U \varepsilon \circ \mu U = 1_U$$

$$\mu \circ T \eta = (U \varepsilon F) \circ (U F \eta) = U(\varepsilon F \circ F \eta) = U 1_F = 1_{UF}$$

$$\mu \circ \eta T = (U \varepsilon F) \circ (\eta U F) = (U \varepsilon \circ \eta U) F = 1_U F = 1_{UF}$$

$$\begin{aligned} (U \varepsilon F) \circ (\eta U F) &= (U(\varepsilon(F(b))))(U F(\eta(b))) \\ &= U(\varepsilon(F(b)) F(\eta(b))) = U(1_F) \end{aligned}$$

$$\begin{array}{ccccc} UFUFUF & \xrightarrow{U \varepsilon F U F} & UFUF & & FUFUF & \xrightarrow{\varepsilon F U F} & FUF & & FU(a) & \xrightarrow{\varepsilon(a)} & a \\ \downarrow UFU \varepsilon F & & \downarrow U \varepsilon F & & \downarrow FU \varepsilon F & & \downarrow \varepsilon F & & \downarrow FU(f) & & \downarrow f \\ UFUF & \xrightarrow{U \varepsilon F} & UF & & FUF & \xrightarrow{\varepsilon F} & F & & FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

association law

$$\begin{aligned} \mu \circ \mu T &= \mu \circ T \mu \\ U \varepsilon(a) F \circ U \varepsilon(a) F F U &= U \varepsilon(a) F \circ F U U \varepsilon(a) F \end{aligned}$$

$$U \varepsilon F \circ U \varepsilon F F U = U \varepsilon F \circ F U U \varepsilon F$$

naturality of  $\varepsilon$

$$\varepsilon(b) F U(f)(a) = f \varepsilon(a)$$

$$a = FUF(a), b = F(a), f = \varepsilon F$$

$$\begin{aligned} \varepsilon(F(a))(FU(\varepsilon F))(a) &= (\varepsilon F)(\varepsilon FUF(a)) \\ U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\ U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\ U\varepsilon F \circ FUU\varepsilon F &= U\varepsilon F \circ U\varepsilon FFU \end{aligned}$$

$$\begin{array}{ccc} FUF(a) & \xleftarrow{FU(\varepsilon(F(a)))} & FUFUF(a) \\ \downarrow \varepsilon(F(a)) & & \downarrow \varepsilon(FUF(a)) \\ F(a) & \xleftarrow{\varepsilon(F(a))} & FUF(a) \end{array}$$

## 7 Eilenberg-Moore category

$(T, \eta, \mu)$

$A^T$  object  $(A, \phi)$

$\phi\eta(A) = 1_A, \phi\mu(A) = \phi T(\phi)$

arrow  $f$ .

$\phi T(f) = f\phi$

$$\begin{array}{ccc} a \xrightarrow{\eta(a)} T(a) & T^2(a) \xrightarrow{\mu(a)} T(a) & T(a) \xrightarrow{T(f)} T(b) \\ \searrow 1_a \quad \downarrow \phi & \downarrow T(\phi) \quad \downarrow \phi & \downarrow \phi \quad \downarrow \phi' \\ a & T(a) \xrightarrow{\phi} T(a) & a \xrightarrow{f} b \end{array}$$

## 8 EM on monoid

$f : a \rightarrow b$

$T : a \rightarrow (m, a)$

$$\begin{aligned}\eta &: a \rightarrow (1, a) \\ \mu &: (m, (m', a)) \rightarrow (mm', a) \\ \phi &: (m, a) \rightarrow \phi(m, a) = ma\end{aligned}$$

$$\begin{array}{ccc} a \xrightarrow{\eta(a)} (1, a) & (m, (m', a)) \xrightarrow{\mu(a)} (mm', a) & (m, a) \xrightarrow{T(f)} (m, f(a)) \\ \searrow 1_a \downarrow \phi & \downarrow T(\phi) \downarrow \phi & \downarrow \phi \downarrow \phi' \\ & (m, m'a) \xrightarrow{\phi} mm'a & ma \xrightarrow{f} mf(a) = f(ma)\end{array}$$

object  $(a, \phi)$ . arrow  $f$ .

$$\begin{aligned}\phi T(f)(m, a) &= f\phi(m, a) \\ \phi(m, f(a)) &= f(a) \\ U^T : A^T &\rightarrow A \\ U^T(a, \phi) &= a, U^T(f) = f \\ F^T : A &\rightarrow A^T \\ F^T(a) &= (T(a), \mu(a)), F^T(f) = T(f)\end{aligned}$$

## 9 Comparison Functor $K^T$

$$\begin{aligned}K^T(B) &= (U(B), U\varepsilon(B))a, K^T(f) = U(f) \\ U^T K^T(b) &= U(b) \\ U^T K^T(f) &= U^T U(f) = U(f) \\ K^T F(a) &= (UF(a), U\varepsilon(F(a))) = (T(a), \mu(a)) = F^T(a) \\ K^T F(f) &= UF(f) = T(f) = F^T(f) \\ \eta U(a, \phi) &= \eta(a), U\varepsilon(a, \phi) = \varepsilon^T K^T(b) = U\varepsilon(b)\end{aligned}$$

$$\begin{array}{ccc}
B & \xrightarrow{K^T} & A^T \\
\uparrow K_T & \swarrow F & \uparrow U^T \\
& & A \\
A_T & \xleftarrow{U_T} & A
\end{array}$$

$$\begin{array}{ccccc}
& & B & & \\
& \nearrow K_T & \uparrow U & \searrow K^T & \\
A_T & \xrightarrow{U_T} & A & \xrightarrow{F^T} & A^T
\end{array}$$

## 10 Kleisli Category

Object of  $A$ .

Arrow  $f : a \rightarrow T(a)$  in  $A$ . In  $A_T$ ,  $f : b \rightarrow c, g : c \rightarrow d$ ,

$$g * f = \mu(d)T(g)f$$

$\eta(b) : b \rightarrow T(b)$  is an identity.

$$f * \eta(b) = \mu(c)T(f)\eta(b) = \mu(c)\eta(T(c))f = 1_T(c)f = f$$

and

$$\eta(c) * f = \mu(c)T\eta(c)f = 1_T(c)f = f$$

association law  $g * (f * h) = (g * f) * h$ ,

$h : a \rightarrow T(b), f : b \rightarrow T(c), g : c \rightarrow T(d)$ ,

naturality of  $\mu$



$$\begin{array}{lcl}
f * h & & T(c) \xleftarrow{\mu(c)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a \\
g * (f * h) & & T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T(g)} T(c) \xleftarrow{\mu(c)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a \\
(g * f) * h & & T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{\mu(d)T} T^3(d) \xleftarrow{T^2(g)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a \\
(g * f) * h & & T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T\mu(d)} T^3(d) \xleftarrow{T^2(g)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a \\
g * f & & T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T(g)} T(c) \xleftarrow{f} b
\end{array}$$

$$\begin{array}{ccccc}
T^2(d) & \xleftarrow{T\mu(d)} & T^2(T(d)) & \xleftarrow{T^2(g)} & T^2(c) \\
\mu(d) \downarrow & & \downarrow \mu(T(d)) & & \downarrow \mu(c) \\
T(d) & \xleftarrow{\mu(d)} & T(T(d)) & \xleftarrow{T(g)} & T(c)
\end{array}$$

$$\begin{aligned}
g * (f * h) &= g * (\mu(c)T(f)h) \\
&= \mu(d)(T(g))(\mu(c)T(f)h) \\
&= \mu(d)T(g)\mu(c)T(f)h
\end{aligned}$$

$$\begin{aligned}
(g * f) * h &= (\mu(d)T(g)f) * h \\
&= \mu(d)T(\mu(d)T(g)f)h \\
&= \mu(d)T(\mu(d))T^2(g)T(f)h
\end{aligned}$$

$$\begin{aligned}
\mu(d)\mu(d)T &= \mu(d)T\mu(d) \\
\mu(T(d))T^2(g) &= T(g)\mu(c) \text{ naturality of } \mu. \\
\mu(d)T\mu(d)T^2(g) &= \mu(d)\mu(T(d))T^2(g) = \mu(d)T(g)\mu(c)
\end{aligned}$$

$$\begin{array}{ccc}
T^3(d) & \xleftarrow{T^2(g)} & T^2(c) \\
\downarrow \mu(T(d)) & & \uparrow \mu(c) \\
T^2(d) & \xleftarrow{T(g)} & T(c)
\end{array}$$

$\mu(T(d)) = T\mu(d)?$   
 $(m, (m'm'', a)) = (mm', (m'', a))$  No, but  
 $\mu\mu(T(d)) = \mu T\mu(d).$

## 11 Ok

$T(g)\mu(c) = T(\mu(d))T^2(g)$  であれば良いが。  
 $\mu(d)T^2(g) = T(g)\mu(c)$   
 ちょっと違う。  $\mu(d)T(\mu(d))T^2(g)$  が、  
 $\mu(d)\mu(d)T^2(g)$   
 となると良いが。  
 $\mu(d)T(\mu(d)) = \mu(d)\mu(T(d))$

## 12 monoid in Kleisli category

$T : a \rightarrow (m, a)$   
 $T : f \rightarrow ((m, a) \rightarrow (m, f(a)))$   
 $\mu(a) : (m, (m', a)) \rightarrow (mm', a)$   
 $f : a \rightarrow (m, f(a))$

$$\begin{aligned}
g * f(b) &= \mu(d)T(g)f(b) = \mu(d)T(g)(m, f(b)) \\
&= \mu(m, (m', gf(b))) = (mm', gf(b)) \\
(g * f) * h(a) &= \mu(d)T(\mu(d)T(g)f)h(a) = \mu(d)T(\mu(d))(TT(g))T(f)(m, h(a)) \\
&= \mu(d)T(\mu(d))(TT(g))(m, (m', fh(a))) \\
&= \mu(d)T(\mu(d)(m, (m', (m'', gfh(a)))) = (mm'm'', gfh(a)) \\
g * (f * h)(a) &= (\mu(d)(T(g)))\mu(c)T(f)h(a) = (\mu(d)(T(g)))\mu(c)T(f)(m, h(a)) \\
&= (\mu(d)(T(g)))\mu(c)(m, (m', fh(a)))
\end{aligned}$$

$$= \mu(d)T(g)(mm', fh(a)) = (mm'm'', gfh(a))$$

### 13 Resolution of Kleiseli category

$$f : a \rightarrow b, g : b \rightarrow c$$

$$U_T : A_T \rightarrow A$$

$$U_T(a) = T(a)$$

$$U_T(f) = \mu(b)T(f)$$

$$g * f = \mu(d)T(g)f$$

$$U_T(g * f) = U_T(\mu(c)T(g)f)$$

$$= \mu(c)T(\mu(c)T(g)f)$$

$$= \mu(c)\mu(c)T(T(g)f) = \mu(c)\mu(c)TT(g)T(f) \text{ association law}$$

$$U_T(g)U_T(f) = \mu(c)T(g)\mu(b)T(f) = \mu(c)\mu(c)TT(g)T(f)$$

$$T(g)\mu(b) = \mu(c)TT(g)$$

$$\begin{array}{ccc} TT & \xleftarrow{TT(g)} & TT \\ \downarrow \mu(c) & & \downarrow \mu(b) \\ T & \xleftarrow{T(g)} & T \end{array}$$

$$F_T : A \rightarrow A_T$$

$$F_T(a) = a$$

$$F_T(f) = \eta(b)f$$

$$F_T(1_a) = \eta(a) = 1_{F_T(a)}$$

$$\begin{aligned} F_T(g) * F_T(f) &= \mu(c)T(F_T(g))F_T(f) \\ &= \mu(c)T(\eta(c)g)\eta(b)f \\ &= \mu(c)T(\eta(c))T(g)\eta(b)f \\ &= T(g)\eta(b)f \text{ unity law} \\ &= \eta(c)gf = F_T(gf) \end{aligned}$$

$$\eta(c)g = T(g)\eta(b)$$

$$\begin{array}{ccc} c & \xleftarrow{g} & b \\ \downarrow \eta(c) & & \downarrow \eta(b) \\ T & \xleftarrow{T(g)} & T \end{array}$$

$$\mu \circ T\eta = 1_T = \mu \circ \eta T \text{ Unity law}$$

$$\varepsilon_T(a) = 1_{T(a)}$$

$$U_T \varepsilon_T F_T = \mu$$

$$U_T \varepsilon_T F_T a(a) = U_T \varepsilon_T(a) = U_T(1_{T(a)}) = \mu(a)$$

$$\begin{aligned} \varepsilon_T(F_T(a)) * F_T(\eta(a)) &= \varepsilon_T(a) * F_T(\eta(a)) \\ &= 1_{T(a)} * (F_T(\eta(a))) \\ &= 1_{T(a)} * (\eta(T(a))\eta(a)) \\ &= \mu(T(a))T(1_{T(a)})(\eta(T(a))\eta(a)) \\ &= \mu(T(a))\eta(T(a))\eta(a) \\ &= \eta(a) = 1_{F_T} \end{aligned}$$

$$\begin{aligned} U_T(\varepsilon_T(a))\eta(U_T(a)) &= U_T(1_{T(a)}\eta(T(a))) \\ &= \mu(T(a))T(1_{T(a)})\eta(T(a)) \\ &= \mu(T(a))\eta(T(a))1_{T(a)} \\ &= 1_{T(a)} = T(1_a) = 1_{U_T} \end{aligned}$$

### 13.1 Comparison functor $K_T$

Adjoint  $(B, U, F, \varepsilon)$ ,  $K_T : A_T \rightarrow B$ ,  
 $g : b \rightarrow c$ .

$$\begin{aligned}
 K_T(a) &= F(a) \\
 K_T(g) &= \varepsilon(F(c))F(g) \\
 K_T F_T(a) &= K_T(a) = F(a) \\
 K_T F_T(f) &= K_T(\eta(b)f) \\
 &= \varepsilon(F(b))F(\mu(b)f) \\
 &= \varepsilon(F(b))F(\mu(b))F(f) = F(f)
 \end{aligned}$$

$$\begin{aligned}
 K_T(\eta(b)) &= \varepsilon(F(b))F(\eta(b)) = 1_{F(b)} \\
 K_T(\eta(T(c))g) &= \varepsilon(F(T(c)))F(\eta(T(c))g) = F(g) \\
 K_T(g)K_T(f) &= \varepsilon(F(c))F(g)\varepsilon(F(b))F(f) = \varepsilon(F(c))\varepsilon(F(c))FUF(g)F(f) \\
 K_T(g * f) &= \varepsilon(F(c))F(\mu(c)UF(g)f) = \varepsilon(F(c))F(\mu(c))FUF(g)F(f) \\
 \varepsilon(F(c))FUF(g) &= F(g)\varepsilon(F(b))
 \end{aligned}$$

$$\begin{array}{ccc}
 FU(F(c)) & \xleftarrow{FU(F(g))} & FU(F(b)) \\
 \downarrow \varepsilon(F(c)) & & \downarrow \varepsilon(F(b)) \\
 F(c) & \xleftarrow{F(g)} & F(b)
 \end{array}$$

$$\begin{aligned}
 \varepsilon(F(c))F(\mu(c)) &= \varepsilon(F(c))\varepsilon(F(c)) ? \\
 \varepsilon(F(c))F(\mu(c)) &= \varepsilon(F(c))FU\varepsilon(F(c))
 \end{aligned}$$

$$\begin{array}{ccc}
FUFU(c) & \xleftarrow{FU\varepsilon(F(c))} & FUFU(F(c)) \\
\downarrow \varepsilon F(c) & & \downarrow \varepsilon(F(c)) \\
FU(c) & \xleftarrow{\varepsilon(F(c))} & FU(F(c))
\end{array}$$

$$\begin{aligned}
UK_T(a) &= UF(a) = T(a) = U_T(a) \\
UK_T(g) &= U(\varepsilon((F(c))F(g))) = U(\varepsilon(F(c)))UF(g) = \mu(c)T(g) = U_T(g)
\end{aligned}$$

## 14 Monoid

$$\begin{aligned}
T : A &\rightarrow MxA \\
T(a) &= (m, a) \\
T(f) : T(A) &\rightarrow T(f(A)) \\
T(f)(m, a) &= (m, f(a)) \\
T(fg)(m, a) &= (m, fg(a))
\end{aligned}$$

## 15 association of Functor

$$\begin{aligned}
T(f)T(g)(m, a) &= T(f)(m, g(a)) = (m, fg(a)) = T(fg)(m, a) \\
\mu : TxT &\rightarrow T \\
\mu_a(T(T(a))) &= \mu_A((m, (m', a))) = (m * m', a)
\end{aligned}$$

## 16 TT

$$\begin{aligned}
TT(a) &= (m, (m', a)) \\
TT(f)(m, (m', a)) &= (m, (m', f(a)))
\end{aligned}$$

## 17 naturality of $\mu$

$$\begin{array}{ccc}
 TT(a) & \xrightarrow{\mu(a)} & T(a) \\
 \downarrow TT(f) & & \downarrow T(f) \\
 TT(b) & \xrightarrow{\mu(b)} & T(b)
 \end{array}$$

$$\begin{aligned}
 \mu(b)TT(f)TT(a) &= T(f)\mu(a)TT(a) \\
 \mu(b)TT(f)TT(a) &= \mu(b)((m, (m', f(a)))) = (m * m', f(a)) \\
 T(f)\mu(a)(TT(a)) &= T(f)(m * m', a) = (m * m', f(a))
 \end{aligned}$$

## 18 $\mu$ $\mu$

Horizontal composition of  $\mu$

$$\begin{array}{ccc}
 f & \rightarrow & \mu_T T(a) \\
 a & \rightarrow & TT(a) \\
 \mu_T(a)TTT(a) & = & \mu_T(a)(m, (m', (m'', a))) = (m * m', (m'', a)) \\
 TTTT(a) & \xrightarrow{\mu(TTT(a))} & TTT(a) \\
 TT(\mu) \downarrow & & \downarrow T(\mu) \\
 TTT(a) & \xrightarrow{\mu(TT(a))} & TT(a)
 \end{array}$$

$$\begin{aligned}
 T(\mu_a)\mu_a TTTT(a) &= T(\mu_a)\mu_a(m_0, (m_1, (m_2, (m_3, a)))) \\
 &= T(\mu_a)(m_0 * m_1, (m_2, (m_3, a))) = (m_0 * m_1, (m_2 * m_3, a)) \\
 \mu_b TT(\mu_a) TTTT(a) &= \mu_b TT(\mu_a)(m_0, (m_1, (m_2, (m_3, a)))) \\
 &= \mu_b(m_0, (m_1, (m_2 * m_3, a))) = (m_0 * m_1, (m_2 * m_3, a))
 \end{aligned}$$

Horizontal composition of natural transformation

## 19 Natural transformation $\varepsilon$ and Functor $F$ : $A \rightarrow B, U : B \rightarrow A$

$$\varepsilon : FUFU \rightarrow FU$$

$$\varepsilon : FU \rightarrow 1_B$$

Naturality of  $\varepsilon$

$$\begin{array}{ccc} FU(a) & \xrightarrow{\varepsilon(a)} & a \\ \downarrow FU(f) & & \downarrow f \\ FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

$$\varepsilon(b)FU(f)a = f\varepsilon(a)a$$

$$\begin{array}{ccc} FUFU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) \\ \downarrow FUFU(f) & & \downarrow FU(f) \\ FUFU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) \end{array}$$

$$\varepsilon((FU(b))FUFU(f)FU(a)) = FU(f)\varepsilon(FU(a))FU(a)$$

## 20 Vertial Compositon $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FUFU \rightarrow 1_B$$

$$\begin{array}{ccccc} FUFU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\ \downarrow FUFU(f) & & \downarrow FU(f) & & \downarrow f \\ FUFU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

## 21 Vertial Compositon $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FUFU \rightarrow 1_B$$



$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 22 Vertical Composition $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FU FU \rightarrow 1B$$

$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 23 Vertical Composition $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FU FU \rightarrow 1B$$

$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 24 Vertical Composition $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FU FU \rightarrow 1B$$

$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 25 Vertical Composition $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FU FU \rightarrow 1B$$

$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 26 Vertical Composition $\varepsilon \bullet \varepsilon$

$$\varepsilon \bullet \varepsilon : FU FU \rightarrow 1_B$$

$$\begin{array}{ccccc}
FU FU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\
\downarrow FU FU(f) & & \downarrow FU(f) & & \downarrow f \\
FU FU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

## 27 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FU FU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc}
FU & & FU \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
1_B & & 1_B
\end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FU FU, FU$  has objects of  $B$ .

$$\varepsilon \circ \varepsilon : FU FU \rightarrow 1_B 1_B$$

$$\begin{array}{ccc}
FU FU(b) & \xrightarrow{\varepsilon^{FU(b)}} & 1_A \\
\downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\
FU 1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B
\end{array}$$

that is

$$\begin{array}{ccc}
FU FU(b) & \xrightarrow{\varepsilon^{FU(b)}} & FU(b) \\
\downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\
FU(b) & \xrightarrow{\varepsilon(b)} & b
\end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

## 28 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\begin{array}{ccc} \varepsilon \circ \varepsilon : FUFU \rightarrow 1_B 1_B & & \\ FUFU(b) \xrightarrow{\varepsilon_{FU(b)}} 1_A & & \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) \xrightarrow{\varepsilon(b)} 1_B & & \end{array}$$

that is

$$\begin{array}{ccc} FUFU(b) \xrightarrow{\varepsilon_{FU(b)}} FU(b) & & \\ \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\ FU(b) \xrightarrow{\varepsilon(b)} b & & \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

## 29 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\varepsilon \circ \varepsilon : FUFU \rightarrow 1_B 1_B$$

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon^{FU(b)}} & 1_A \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B \end{array}$$

that is

$$\begin{array}{ccc}
 FU(b) & \xrightarrow{\varepsilon_{FU(b)}} & FU(b) \\
 \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\
 FU(b) & \xrightarrow{\varepsilon(b)} & b
 \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

### 30 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\begin{array}{ccc} \varepsilon \circ \varepsilon : FUFU \rightarrow 1_B 1_B & & \\ FUFU(b) \xrightarrow{\varepsilon_{FU(b)}} 1_A & & \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) \xrightarrow{\varepsilon(b)} 1_B & & \end{array}$$

that is

$$\begin{array}{ccc} FUFU(b) \xrightarrow{\varepsilon_{FU(b)}} FU(b) & & \\ \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\ FU(b) \xrightarrow{\varepsilon(b)} b & & \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

### 31 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\varepsilon \circ \varepsilon : FUFU \rightarrow 1_B 1_B$$

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon^{FU(b)}} & 1_A \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B \end{array}$$



that is

$$\begin{array}{ccc}
 FU(b) & \xrightarrow{\varepsilon_{FU(b)}} & FU(b) \\
 \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\
 FU(b) & \xrightarrow{\varepsilon(b)} & b
 \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

## 32 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\begin{array}{ccc} \varepsilon \circ \varepsilon : FUFU & \rightarrow & 1_B 1_B \\ FUFU(b) & \xrightarrow{\varepsilon_{FU(b)}} & 1_A \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B \end{array}$$

that is

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon_{FU(b)}} & FU(b) \\ \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\ FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$$

remove  $FU(b)$  on right,

$$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$$

this shows commutativity of previous diagram

$$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$$

that is

$$\varepsilon\varepsilon FU = \varepsilon FU \varepsilon$$

### 33 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & & FU \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf.  $FUFU, FU$  has objects of  $B$ .

$$\varepsilon \circ \varepsilon : FUFU \rightarrow 1_B 1_B$$

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon^{FU(b)}} & 1_A \\ \downarrow FU\varepsilon(b) & & \downarrow 1_a\varepsilon(b) \\ FU1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B \end{array}$$

that is

$$\begin{array}{ccc}
 FU(b) & \xrightarrow{\varepsilon_{FU(b)}} & FU(b) \\
 \downarrow FU\varepsilon(b) & & \downarrow \varepsilon(b) \\
 FU(b) & \xrightarrow{\varepsilon(b)} & b
 \end{array}$$

$\varepsilon(b) : b \rightarrow \varepsilon(b)$  arrow of  $B$

$\varepsilon : FU \rightarrow 1_B$

$\varepsilon(b) : FU(b) \rightarrow b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace  $f$  by  $\varepsilon(b)$ ,  $a$  by  $FU(b)$  in naturality  $\varepsilon(b)FU(f)a = f\varepsilon(a)a$

$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$

remove  $FU(b)$  on right,

$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$

this shows commutativity of previous diagram

$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$

that is

$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

## 34 Yoneda Functor

$Y : A \rightarrow Sets^{A^{op}}$

$Hom_A : A^{op} \times A \rightarrow Sets$

$g : a' \rightarrow a, h : b \rightarrow b'$

$Hom_A((g, h)) : Home_A(a, b) \rightarrow \{hfg | f \in Home_A(a, b)\}$

$Hom_A((g, h) \circ (g', h')) : Home_A(a, b) \rightarrow \{hh'fgg' | f \in Home_A(a, b)\}$

$Hom_A((g, h))Hom_A((g', h')) : Home_A(a, b) \rightarrow \{h'fg' | f \in Home_A(a, b)\} \rightarrow \{hh'fgg' | f \in Home_A(a, b)\}$

$$\begin{array}{ccccc}
a & \xrightarrow{g'} & a' & \xrightarrow{g} & a'' \\
& & \downarrow & & \downarrow f \\
b & \xleftarrow{h'} & b' & \xleftarrow{h} & b''
\end{array}$$

$Hom_A^* : A^{op} \rightarrow Sets^A$   
 $f^{op} : a \rightarrow c (f : c \rightarrow a)$   
 $g^{op} : c \rightarrow d (g : d \rightarrow c)$   
 $Home_A^*(a) : a \rightarrow \lambda b. Hom_A(a, b)$   
 $Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b))$   
 $Home_A^*(g^{op} f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (d \rightarrow \lambda b. Hom_A(fg(d), b))$   
 $Home_A^*(g^{op}) Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b)) \rightarrow$   
 $(d \rightarrow \lambda b. Hom_A(fg(d), b))$   
 $Hom_{A^{op}}^* : A \rightarrow Sets^{A^{op}}$   
 $f : c \rightarrow b \quad g : d \rightarrow c$   
 $Home_{A^{op}}^*(b) : b \rightarrow \lambda a. Hom_{A^{op}}(a, b)$   
 $Home_{A^{op}}^*(f) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c)))$   
 $Home_{A^{op}}^*(gf) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (d \rightarrow \lambda a. Hom_{A^{op}}(a, gf(d)))$   
 $Home_{A^{op}}^*(g) Home_{A^{op}}^*(f) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c))) \rightarrow$   
 $(d \rightarrow \lambda a. Hom_{A^{op}}(a, gf(d)))$   
Arrows in  $Set^{A^{op}}$ ?  
 $f : b \rightarrow c = (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c)))$   
 $Set^{A^{op}} : A^{op} \rightarrow Set$   
an object  $b = \lambda a. Hom_{A^{op}}(a, b)$  is a functor from  $A^{op}$  to  $Set$ .  
 $t : (\lambda a. Hom_{A^{op}}(a, b)) \rightarrow (\lambda a. Hom_{A^{op}}(a, t(c)))$  should be a natural trans-  
formatin.

$$\begin{array}{ccc}
f^{op} : (b : A^{op}) \rightarrow (c : A^{op}) = f : c \rightarrow b \\
Hom_{A^{op}}(a, c) \xrightarrow{t(c)} Hom_{A^{op}}(a, t(c)) \\
\downarrow Home^* A^{op}(a, f) \quad \quad \quad \uparrow Home^* A^{op}(a, f) \\
Hom_{A^{op}}(a, b) \xrightarrow{t(b)} Hom_{A^{op}}(a, t(b))
\end{array}$$

### 34.1 Contravariant functor

$$h_a = \text{Hom}_A(-, a)$$

$$f : b \rightarrow c, \text{Hom}_A(f, 1_a) : \text{Hom}_A(c, a) \rightarrow \text{Hom}_A(b, a)$$

## 35 Yoneda Functor

$$Y : A \rightarrow \text{Sets}^{A^{op}}$$

$$\text{Hom}_A : A^{op} \times A \rightarrow \text{Sets}$$

$$g : a' \rightarrow a, h : b \rightarrow b'$$

$$\text{Hom}_A((g, h)) : \text{Home}_A(a, b) \rightarrow \{hfg | f \in \text{Home}_A(a, b)\}$$

$$\text{Hom}_A((g, h) \circ (g', h')) : \text{Home}_A(a, b) \rightarrow \{hh'fgg' | f \in \text{Home}_A(a, b)\}$$

$$\text{Hom}_A((g, h))\text{Hom}_A((g', h')) : \text{Home}_A(a, b) \rightarrow \{h'fg' | f \in \text{Home}_A(a, b)\} \rightarrow \{hh'fgg' | f \in \text{Home}_A(a, b)\}$$

$$\begin{array}{ccccc} a & \xrightarrow{g'} & a' & \xrightarrow{g} & a'' \\ & & \downarrow & & \downarrow f \\ & & b' & \xleftarrow{h} & b'' \\ b & \xleftarrow{h'} & & & \end{array}$$

$$\text{Hom}_A^* : A^{op} \rightarrow \text{Sets}^A$$

$$f^{op} : a \rightarrow c (f : c \rightarrow a)$$

$$g^{op} : c \rightarrow d (g : d \rightarrow c)$$

$$\text{Home}_A^*(a) : a \rightarrow \lambda b. \text{Hom}_A(a, b)$$

$$\text{Home}_A^*(f^{op}) : (a \rightarrow \lambda b. \text{Hom}_A(a, b)) \rightarrow (c \rightarrow \lambda b. \text{Hom}_A(f(c), b))$$

$$\text{Home}_A^*(g^{op} f^{op}) : (a \rightarrow \lambda b. \text{Hom}_A(a, b)) \rightarrow (d \rightarrow \lambda b. \text{Hom}_A(fg(d), b))$$

$$\text{Home}_A^*(g^{op})\text{Home}_A^*(f^{op}) : (a \rightarrow \lambda b. \text{Hom}_A(a, b)) \rightarrow (c \rightarrow \lambda b. \text{Hom}_A(f(c), b)) \rightarrow (d \rightarrow \lambda b. \text{Hom}_A(fg(d), b))$$

$$\text{Hom}_{A^{op}}^* : A \rightarrow \text{Sets}^{A^{op}}$$

$$f : c \rightarrow b \quad g : d \rightarrow c$$

$$\text{Home}_{A^{op}}^*(b) : b \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, b)$$

$$\text{Home}_{A^{op}}^*(f) : (b \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, f(c)))$$

$$\text{Home}_{A^{op}}^*(gf) : (b \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, b)) \rightarrow (d \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, gf(d)))$$

$$\text{Home}_{A^{op}}^*(g)\text{Home}_{A^{op}}^*(f) : (b \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, f(c))) \rightarrow (d \rightarrow \lambda a. \text{Hom}_{A^{op}}(a, gf(d)))$$

$$\text{Arrows in } \text{Set}^{A^{op}}?$$

$$f : b \rightarrow c = (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c)))$$

$$Set^{A^{op}} : A^{op} \rightarrow Set$$

an object  $b = \lambda a. Hom_{A^{op}}(a, b)$  is a functor from  $A^{op}$  to  $Set$ .

$t : (\lambda a. Hom_{A^{op}}(a, b)) \rightarrow (\lambda a. Hom_{A^{op}}(a, t(c)))$  should be a natural transformation.

$$f^{op} : (b : A^{op}) \rightarrow (c : A^{op}) = f : c \rightarrow b$$

$$\begin{array}{ccc} Hom_{A^{op}}(a, c) & \xrightarrow{t(c)} & Hom_{A^{op}}(a, t(c)) \\ \downarrow Home^* A^{op}(a, f) & & \uparrow Home^* A^{op}(a, f) \\ Hom_{A^{op}}(a, b) & \xrightarrow{t(b)} & Hom_{A^{op}}(a, t(b)) \end{array}$$

### 35.1 Contravariant functor

$$h_a = Hom_A(-, a)$$

$$f : b \rightarrow c, Hom_A(f, 1_a) : Hom_A(c, a) \rightarrow Hom_A(b, a)$$

## 36 Yoneda Functor

$$Y : A \rightarrow Sets^{A^{op}}$$

$$Hom_A : A^{op} \times A \rightarrow Sets$$

$$g : a' \rightarrow a, h : b \rightarrow b'$$

$$Hom_A((g, h)) : Home_A(a, b) \rightarrow \{hfg | f \in Home_A(a, b)\}$$

$$Hom_A((g, h) \circ (g', h')) : Home_A(a, b) \rightarrow \{hh'fgg' | f \in Home_A(a, b)\}$$

$$Hom_A((g, h))Hom_A((g', h')) : Home_A(a, b) \rightarrow \{h'fg' | f \in Home_A(a, b)\} \rightarrow \{hh'fgg' | f \in Home_A(a, b)\}$$

$$\begin{array}{ccccc} a & \xrightarrow{g'} & a' & \xrightarrow{g} & a'' \\ & & \downarrow & & \downarrow f \\ b & \xleftarrow{h'} & b' & \xleftarrow{h} & b'' \end{array}$$

$$Hom_A^* : A^{op} \rightarrow Sets^A$$

$$f^{op} : a \rightarrow c(f : c \rightarrow a)$$

$$\begin{aligned}
&g^{op} : c \rightarrow d (g : d \rightarrow c) \\
&Home_A^*(a) : a \rightarrow \lambda b. Hom_A(a, b) \\
&Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b)) \\
&Home_A^*(g^{op} f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (d \rightarrow \lambda b. Hom_A(fg(d), b)) \\
&Home_A^*(g^{op}) Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b)) \rightarrow \\
&(d \rightarrow \lambda b. Hom_A(fg(d), b)) \\
&Hom_{A^{op}}^* : A \rightarrow Sets^{A^{op}} \\
&f : c \rightarrow b \quad g : d \rightarrow c \\
&Home_{A^{op}}^*(b) : b \rightarrow \lambda a. Hom_{A^{op}}(a, b) \\
&Home_{A^{op}}^*(f) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c))) \\
&Home_{A^{op}}^*(gf) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (d \rightarrow \lambda a. Hom_{A^{op}}(a, gf(d))) \\
&Home_{A^{op}}^*(g) Home_{A^{op}}^*(f) : (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c))) \rightarrow \\
&(d \rightarrow \lambda a. Hom_{A^{op}}(a, gf(d))) \\
&\text{Arrows in } Set^{A^{op}}?
\end{aligned}$$

$$f : b \rightarrow c = (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c)))$$

$$Set^{A^{op}} : A^{op} \rightarrow Set$$

an object  $b = \lambda a. Hom_{A^{op}}(a, b)$  is a functor from  $A^{op}$  to  $Set$ .

$t : (\lambda a. Hom_{A^{op}}(a, b)) \rightarrow (\lambda a. Hom_{A^{op}}(a, t(c)))$  should be a natural transformation.

$$\begin{array}{ccc}
f^{op} : (b : A^{op}) \rightarrow (c : A^{op}) = f : c \rightarrow b & & \\
Hom_{A^{op}}(a, c) \xrightarrow{t(c)} Hom_{A^{op}}(a, t(c)) & & \\
\downarrow Home^*_{A^{op}}(a, f) & & \uparrow Home^*_{A^{op}}(a, f) \\
Hom_{A^{op}}(a, b) \xrightarrow{t(b)} Hom_{A^{op}}(a, t(b)) & & 
\end{array}$$

## 36.1 Contravariant functor

$$h_a = Hom_A(-, a)$$

$$f : b \rightarrow c, Hom_A(f, 1_a) : Hom_A(c, a) \rightarrow Hom_A(b, a)$$

## 37 Yoneda Functor

$$Y : A \rightarrow Sets^{A^{op}}$$

$$Hom_A : A^{op} \times A \rightarrow Sets$$



$$\begin{aligned}
& g : a' \rightarrow a, h : b \rightarrow b' \\
& Hom_A((g, h)) : Home_A(a, b) \rightarrow \{hfg \mid f \in Home_A(a, b)\} \\
& Hom_A((g, h) \circ (g', h')) : Home_A(a, b) \rightarrow \{hh'fgg' \mid f \in Home_A(a, b)\} \\
& Hom_A((g, h))Hom_A((g', h')) : Home_A(a, b) \rightarrow \{h'fg' \mid f \in Home_A(a, b)\} \rightarrow \\
& \{hh'fgg' \mid f \in Home_A(a, b)\}
\end{aligned}$$

$$\begin{array}{ccccc}
a & \xrightarrow{g'} & a' & \xrightarrow{g} & a'' \\
& & \downarrow & & \downarrow f \\
b & \xleftarrow{h'} & b' & \xleftarrow{h} & b''
\end{array}$$

$$\begin{aligned}
& Hom_A^* : A^{op} \rightarrow Sets^A \\
& f^{op} : a \rightarrow c (f : c \rightarrow a) \\
& g^{op} : c \rightarrow d (g : d \rightarrow c) \\
& Home_A^*(a) : a \rightarrow \lambda b. Hom_A(a, b) \\
& Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b)) \\
& Home_A^*(g^{op} f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (d \rightarrow \lambda b. Hom_A(fg(d), b)) \\
& Home_A^*(g^{op})Home_A^*(f^{op}) : (a \rightarrow \lambda b. Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b. Hom_A(f(c), b)) \rightarrow \\
& (d \rightarrow \lambda b. Hom_A(fg(d), b))
\end{aligned}$$

Arrows in  $Set^{A^{op}}$ ?

$$f : b \rightarrow c = (b \rightarrow \lambda a. Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a. Hom_{A^{op}}(a, f(c)))$$

$$Set^{A^{op}} : A^{op} \rightarrow Set$$

an object  $b = \lambda a. Hom_{A^{op}}(a, b)$  is a functor from  $A^{op}$  to  $Set$ .

$t : (\lambda a. Hom_{A^{op}}(a, b)) \rightarrow (\lambda a. Hom_{A^{op}}(a, t(c)))$  should be a natural transformation.

$$f^{op} : (b : A^{op}) \rightarrow (c : A^{op}) = f : c \rightarrow b$$

$$\begin{array}{ccc}
Hom_{A^{op}}(a, c) & \xrightarrow{t(c)} & Hom_{A^{op}}(a, t(c)) \\
\downarrow Hom_{A^{op}}(a, f) & & \uparrow Hom_{A^{op}}(a, f) \\
Hom_{A^{op}}(a, b) & \xrightarrow{t(b)} & Hom_{A^{op}}(a, t(b))
\end{array}$$

### 37.1 Contravariant functor

$$\begin{aligned}
h_a &= Hom_A(-, a) \\
f : b \rightarrow c, Hom_A(f, 1_a) &: Hom_A(c, a) \rightarrow Hom_A(b, a)
\end{aligned}$$