Cateogry

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1 Adjunction

$$\begin{aligned} U\varepsilon \circ \eta U &= 1_U \\ \varepsilon F \circ F\eta &= 1_F \end{aligned}$$

$f : a \rightarrow Ub$

$$\begin{tikzcd} & FU(b) \arrow[d, "\varepsilon(b)"] \\ F(a) \arrow[ur, "F(f)"] \arrow[r, "f*"'] & b \end{tikzcd}$$

$$\begin{tikzcd} UF(a) \arrow[r, "U(f*)"] & Ub \\ a \arrow[u, "\eta(a)"] \arrow[ur, "f"] & \end{tikzcd}$$

Universal mapping problem is for each $f : \rightarrow Ub$, there exists $f*$ such that $f = U(f*)\eta$.

2 Adjunction to Universal mapping

In adjunction $(F, U, \varepsilon, \eta)$, put $f* = \varepsilon(b)F(f)$, we are going to prove $f*$ is a solution of Universal mmapping problem. That is $U(f*)\eta = f$.

$$\begin{tikzcd}
UF(a) \arrow[r, "UF(f)"'] \arrow[rr, bend left, "U(\varepsilon(b)F(f))=U(f*)"] & UFUb \arrow[r, "U(\varepsilon(b))"'] & ? \\
a \arrow[u, "\eta(a)"] \arrow[r, "f"'] & Ub \arrow[u, "\eta(Ub)"] &
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "UF(f)"] \arrow[dr, "U(f*)"'] & UFUb \arrow[d, bend left, "U(\varepsilon(b))"] \\
a \arrow[u, "\eta(a)"] \arrow[r, "f"'] & Ub \arrow[u, "\eta(Ub)"]
\end{tikzcd}$$

$$\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[dr, "f*"'] & FU(b) \arrow[d, "\varepsilon(b)"] \\
& b
\end{tikzcd}$$

$\because U\varepsilon \circ \eta U = 1_U$

$$U(\varepsilon(b))\eta(U(b)) = 1_{U(b)}$$

means that
$\varepsilon(b) : FU(b) \to b$ is solution of $1_{U(b)}$.
naturality $f\eta(U(b)) = U(F(f))\eta(a)$
gives solution $U(\varepsilon(b))UF(f) = U(F(f)\varepsilon(b))$ for f.

$$U(f*)\eta(a)(a) = f(a)$$

then
$U(\varepsilon(b)F(f))\eta(a)(a) = U(\varepsilon(b))UF(f)\eta(a)(a)$
since F is a functor. And we have

$U(\varepsilon(b))UF(f)\eta(a)(a) = U(\varepsilon(b))\eta(b)f(a)$

because of naturality of η

$$\begin{tikzcd}
UF(a) \arrow[d, "UF(f)"] & a \arrow[l, "\eta(a)"'] \arrow[d, "f"] \\
UF(b) & b \arrow[l, "\eta(b)"']
\end{tikzcd} \qquad UF(f)\eta(a) = \eta(b)f$$

too bad.... we need some thing more.

2.1 Adjoint of η

$U\varepsilon \circ \eta U = 1_U$

$$F(a) \xrightarrow{F(f)} FU(b) \xrightarrow{\varepsilon(b)} b$$

$$UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

$$a \xrightarrow{\eta(a)} UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

$$a \xrightarrow{f} Ub \xrightarrow[\eta(Ub)]{\eta(Ub)} UFU(b) \xrightarrow[U(\varepsilon(b))=1]{U(\varepsilon(b))} U(b)$$

$\because$ $U\varepsilon \circ \eta U = 1_U$

naturality of $f : a \to Ub$

$$\begin{tikzcd}
Ub \arrow[r, "\eta(Ub)"] & UF(Ub) \\
a \arrow[u, "f"] \arrow[r, "\eta(a)"] & UF(a) \arrow[u, "UF(f)"']
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "UF(f)"] & UF(U(b)) & UF(U(b)) \arrow[d, "U(\varepsilon(U(b)))"] \\
a \arrow[u, "\eta(a)"'] \arrow[r, "f"] & U(b) \arrow[u, "\eta(U(b))"'] & U(b)
\end{tikzcd}$$

Solution of universal mapping yields naturality of $U\varepsilon \circ \eta U = 1_U$.
$\varepsilon F \circ F\eta = 1_F$.

$$\begin{tikzcd}
UF(a) \arrow[dr, "1_{UF(a)}"] & \\
a \arrow[u, "\eta(a)"] \arrow[r, "\eta(a)"'] & UF(a)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(a) \arrow[r, "F(\eta(a))"] \arrow[dr, "1_{F(a)}", "(1_{UF(a)})*"'] & FUF(a) \arrow[d, "\varepsilon(F(a))"] \\
 & F(a)
\end{tikzcd}$$

3 Universal mapping to adjunction

Functor U, mapping $F(a)$ and $(f)*, U(f*)\eta(a) = f$ are given.

object $F(a) : A \to B$
$\eta(a) : a \to UF(a)$
put

$$F(f) = (\eta(b)f)*$$

$$\varepsilon : FU \to 1_B$$

$$\varepsilon(b) = (1_{U(b)})*$$

$$\begin{tikzcd}
F(a) \arrow[r, "f*"] & b
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "U(f*)"] & Ub \\
a \arrow[u, "\eta(a)"] \arrow[ur, "f"] &
\end{tikzcd}$$

$f = U(f*)\eta$
Show F is a Functor, that is $F(fg) = F(f)F(g)$.

Show naturality of $\eta(a)$.

$$f : a \to b, F(f) = (\eta(b)f)*$$

Show naturality of $\varepsilon(b) = (1_U)*$.

3.1 Definitions

f's destination

$$f : a \to U(b)$$

universal mapping

$$U(f*)\eta(a) = f$$

defnition of F(f)

$$F(f) = (\eta(U(b))f)*$$

definition of ε

$$\varepsilon(b) = (1_{U(b)})*$$

$$\begin{tikzcd}
FUF(a) \arrow[r, "FU(f*)"] \arrow[d, "\varepsilon(F(a))"] & FU(b) \arrow[d, "\varepsilon(b)=(1_U(b))*"] \\
F(a) \arrow[u, bend left, "F(\eta(a))"] \arrow[ur, swap, "F(f)"] \arrow[r, swap, "f*"] & b
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "UF(f)"] \arrow[dr, swap, "U(f*)"] & UFU(b) \arrow[d, bend left, "U(\varepsilon(b))"] \\
a \arrow[u, "\eta(a)"] \arrow[r, swap, "f"] & U(b) \arrow[u, "\eta(U(b))"]
\end{tikzcd}$$

$\varepsilon F \circ F\eta = 1_F$, $\varepsilon(b) = (1_{U(b)})*$,
$\varepsilon(F(a)) = (1_{UF(a)})*$

$$\begin{tikzcd}
UF(a) \arrow[dr, "U(1_{F(a)})"] & F(a) \arrow[r, "F(\eta(a))"] \arrow[dr, "1_{F(a)}"'] & FUF(a) \arrow[d, "\varepsilon(F(a))"'] \\
a \arrow[u, "\eta(a)"] \arrow[r, "\eta(a)"'] & U(F(a)) & F(a) \arrow[u, bend right, "F(\eta(a))"']
\end{tikzcd}$$

3.2 Functor F

$$F(f) = (\eta(b)f)*$$

$$U(F(f))\eta(a) = \eta(b)f$$

show $F(fg) = F(f)F(g)$

$a \xrightarrow{g} Ub \xrightarrow{f} UUc$

$$\begin{aligned} U(F(g))\eta(a) &= \eta(Ub)g \\ U(F(f))\eta(Ub) &= \eta(UUc)f \end{aligned}$$

show

$$U(F(f)F(g))\eta(a) = \eta(UUc)fg$$

then $F(f)F(g) = F(fg)$

$$\begin{aligned} U(F(f)F(g))\eta(a) &= UF(f)UF(g)\eta(a) \\ &= UF(f)\eta(Ub)g \\ &= \eta(UUc)fg \end{aligned}$$

Q.E.D.

$$\begin{tikzcd}
F(a) \arrow[r, "F(g)"] \arrow[dr, "g*"] & FU(b) \arrow[r, "FU(f)"] \arrow[d, "\varepsilon(b)"] \arrow[dr, "f*"] & FUU(c) \arrow[d, "\varepsilon(Uc)"] \\
 & b & U(c)
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "UFg"] \arrow[dr, "U(g*)"] & UFUb \arrow[r, "UFf"] \arrow[dr, "Uf*"] & UFUUc \\
a \arrow[u, "\eta(a)"] \arrow[r, "g"] & Ub \arrow[u, "\eta(Ub)"] \arrow[r, "f"] & UU(c) \arrow[u, "\eta(UUc)"]
\end{tikzcd}$$

3.3 naturality of η

$\eta : 1 \to UB$

$$\begin{tikzcd}
UF(a) \arrow[r, "UF(f)"'] & UFb \\
a \arrow[u, "\eta(a)"] \arrow[r, "f"] & b \arrow[u, "\eta(b)"]
\end{tikzcd}$$

prove $\eta(b)f = UF(f)\eta(a)$

$$\begin{array}{ccll}
\eta(b)f : & & a \to UFb & \\
F(f) & = & (\eta(b)f)* & \text{(definition)} \\
\eta(b)f & = & U(F(f))\eta(a) &
\end{array}$$

Q.E.D.

3.4 naturality of ε

$$\varepsilon : FU \to 1_B$$

$$U : B \to A$$

$\varepsilon(b) = (1_{U(b)})*$
$U(\varepsilon(b))\eta(U(b)) = 1_{U(b)}$

$U(\varepsilon(b))\eta(U(b))U(b) = U(b)$

$$\begin{tikzcd}
FU(b) \arrow[r, "FU(f)"] \arrow[d, "\varepsilon(b)"] & FU(c) \arrow[d, "\varepsilon(c)"] \\
b \arrow[r, "f"] & c
\end{tikzcd}$$

prove $f\varepsilon(b) = \varepsilon(c)FU(f)$

$$f = Ub \to Uc$$

$$\begin{tikzcd}
F(Ub) \arrow[r, "(1_{U(b)})*"] & b \arrow[r, "f"] & c
\end{tikzcd}$$

$$\begin{tikzcd}
UF(Ub) \arrow[r, "U(1_{U(b)})*"] \arrow[dr, "UFU(f)"'] & Ub \arrow[r, "U(f)"] & U(c) \\
& UFUc \arrow[ur, "U(1_U(c)*)"] & \\
Ub \arrow[uu, "\eta(Ub)"] \arrow[uur, "1_{Ub}"'] \arrow[r, "U(f)"] & Uc \arrow[u, "\eta(Uc)"] \arrow[uur, "1_{Uc}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
F(Ub) \arrow[r, "FU(f)"] & FU(c) \arrow[r, "(1_{U(c)})*"] & c
\end{tikzcd}$$

show $\varepsilon(c)FU(f)$ and $f\varepsilon(b)$ are both solution of $(1_{Uc})U(f)(= U(f)(1_{Ub}))$

$$(f\varepsilon(b)))\eta(Ub)Ub = U(f))U(\varepsilon(b))\eta(Ub)Ub$$

$$= U(f)1_{U(b)}Ub = U(f)Ub = Ufb = U(f)(1_{Ub})Ub$$

$$\therefore f\varepsilon(b) = (U(f)(1_{Ub}))*$$

$UFU(f)\eta(Ub) = \eta(Uc)U(f)$ naturality of η

$$U(\varepsilon(c)FU(f))\eta(Ub)Ub = U(\varepsilon(c))UFU(f)\eta(Ub)Ub$$

$$= U(\varepsilon(c))\eta(Uc)U(f)Ub = 1_{U(c)}U(f)Ub = U(f)Ub = U(f)(1_{Ub})Ub$$

$$\because U(\varepsilon(c))\eta(Uc) = 1_U(c)$$

end of proof.

$$\begin{tikzcd}
c & U(c) \arrow[d, "\eta(U(c))"] & U(b) \arrow[l, "U(f)"'] \arrow[d, "\eta(U(b))"'] & b \arrow[lll, bend right, "f"'] \\
FU(c) \arrow[u, "(1_{Uc})*=\varepsilon(c)"] & UFU(c) \arrow[u, bend left, "U(\varepsilon(c))"] & UFU(b) \arrow[l, "UFU(f)"'] \arrow[u, bend right, "U(\varepsilon(b))"'] & FU(b) \arrow[u, "\varepsilon(b)=(1_{Ub})*"'] \arrow[lll, bend left, "FU(f)"]
\end{tikzcd}$$

It also prove

$$U\varepsilon \circ \eta U = 1_U$$

3.5 $U\varepsilon \circ \eta U = 1_U$

$\varepsilon(b) = (1_U(b))*$

that is

$U((1_U(b)*)\eta(U(b)) = 1_U(b)$ $U(\varepsilon(b))\eta(U(b)) = 1_U(b)$

$\therefore U\varepsilon \circ \eta U = 1_U$

3.6 $\varepsilon F \circ F\eta = 1_F$

$\eta(a) = U(1_F(a))\eta(a)$

$\Rightarrow (\eta(a))* = 1_F(a)$ ... (1)

$\varepsilon(F(a)) = (1_U F(a))*$

$\Rightarrow 1_U F(a) = U(\varepsilon(F(a)))\eta(UF(a))$

times $\eta(a)$ from left

$\eta(a) = U(\varepsilon(F(a)))\eta(UF(a))\eta(a)$

$\eta(UF(a)) = UF\eta(a)$ naturality of η

$\eta(a) = U(\varepsilon(F(a)))(UF\eta(a))\eta(a)$

$= U(\varepsilon(F(a)F\eta(a)))\eta(a)$

$\Rightarrow (\eta(a))* = \varepsilon(F(a))F\eta(a)....(2)$

from (1),(2), since $(\eta(a))*$ is unique

$$\varepsilon(F(a))F\eta(a) = 1_F(a)$$

$$\begin{tikzcd}
UF(a) \arrow[r, "F"] & FUF(a) \arrow[r, "U"] \arrow[d, bend left, "\varepsilon(F(a))"] & UFUF(a) \arrow[d, bend left, "U(\varepsilon F(a))"] \\
a \arrow[u, "\eta(a)"] \arrow[r, "F"] & F(a) \arrow[r, "U"] \arrow[u, bend left, "F(\eta(a))"] & UF(a) \arrow[u, bend left, "\eta(UF(a))"] \\
 & F(a) \arrow[r, "U"] \arrow[u, "(\eta(a))*=1_{Fa}"] & UF(a) \arrow[u, "\eta(a)=U(\varepsilon F(a))\eta(UF(a))"']
\end{tikzcd}$$

$$\begin{tikzcd}
UF(a) \arrow[r, "F"] & FUF(a) \arrow[r, "U"] \arrow[d, bend left, "\varepsilon(F(a))"] & UFUF(a) \arrow[d, bend right, "U(\varepsilon F(a))"'] & FUF(a) \arrow[d, bend left, "\varepsilon(F(a))"] \\
a \arrow[u, "\eta(a)"'] \arrow[r, "F"] & F(a) \arrow[r, "U"] \arrow[u, bend left, "F(\eta(a))"] & UF(a) \arrow[u, bend right, "\eta(UF(a))"'] & F(a) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
UUF(a) \arrow[d, bend left, "U(\varepsilon(Fa))"] \\
UF(a) \arrow[u, bend left, "\eta(UF(a))"]
\end{tikzcd}$$

$$U(\varepsilon(F(a)))\eta(UF(a)) = 1_U F(a)$$

$\varepsilon(F(a)) = (1_U F(a))*$

$$\begin{tikzcd}
FA \arrow[r] \arrow[d, "F\eta(A)"] & UFA \arrow[d, "UF\eta A"] \\
FUFA \arrow[r] & UFUFA
\end{tikzcd}$$

$\varepsilon(FA)$ の定義から $U(\varepsilon(FA)) : UFUFA \to UFA$
唯一性から $\varepsilon(F(A)) : FUFA \to FA$ 従って

$$\varepsilon(F(A))F\eta(A) = 1$$

ってなのを考えました。
$U\eta(A') = U(1(FA'))\eta(A')$ より
$\eta(A')* = 1(FA')$、

$U\eta(A') = U(\varepsilon(FA')F\eta(A'))\eta(A')$ より
$\eta(A')* = \varepsilon(FA')F\eta(A')$ から
$1_F = \varepsilon F.F\eta$ は言えました。

後者で η の自然性と ε の定義を使いました。

4 おまけ

$\varepsilon F \circ F\eta = 1_F, U\varepsilon \circ \eta U = 1_U$

$$\begin{tikzcd}
UFU(a) \arrow[d, bend left, "U(\varepsilon(a))"] & FU(a) \arrow[l, "U"'] \arrow[d, "\varepsilon(a)"] \\
U(a) \arrow[u, bend left, "\eta(U(a))"] & (a) \arrow[l, "U"]
\end{tikzcd}$$

$$\begin{tikzcd}
FUF(a) \arrow[d, bend left, "\varepsilon F(a)"] & UF(a) \arrow[l, "F"'] \\
F(a) \arrow[u, bend left, "F\eta(a)"] & (a) \arrow[l, "F"] \arrow[u, "\eta(a)"']
\end{tikzcd}$$

なら、 $FU(\varepsilon(F(a))) = \varepsilon F(a)$?

$$\begin{tikzcd}
UFU(F(a)) \arrow[d, bend left, "U(\varepsilon(F(a)))"] & FU(F(a)) \arrow[l, "U"'] \arrow[d, "\varepsilon(F(a))"] \\
U(F(a)) \arrow[u, bend left, "\eta(U(a))"] & F(a) \arrow[l, "U"]
\end{tikzcd}$$

$$\begin{tikzcd}
FUFU(F(a)) \arrow[d, bend left, "FU(\varepsilon(F(a)))"] & FU(F(a)) \arrow[l, "FU"'] \arrow[d, "\varepsilon(F(a))"] \\
FU(F(a)) \arrow[u, bend left, "F\eta(U(a))"] & F(a) \arrow[l, "FU"]
\end{tikzcd}$$

$$\begin{tikzcd}
FUF(a) \arrow[d, bend left, "\varepsilon F(a)"] & UF(a) \arrow[l, "F"'] \\
F(a) \arrow[u, bend left, "F\eta(a)"] & (a) \arrow[l, "F"] \arrow[u, "\eta(a)"']
\end{tikzcd}$$

5 Monad

(T, η, μ)

$T : A \to A$

$\eta : 1_A \to T$

$\mu : T^2 \to T$

$\mu \circ T\eta = 1_T = \mu \circ \eta T$ Unity law

$\mu \circ \mu T = \mu \circ T\mu$ association law

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

6 Adjoint to Monad

Monad (UF, η , U ε F) on adjoint (U,F, η , ε)

$$\begin{aligned} \varepsilon F \circ F\mu &= 1_F \\ U\varepsilon \circ \mu U &= 1_U \end{aligned}$$

$$\begin{aligned} \mu \circ T\eta &= (U\varepsilon F) \circ (UF\eta) = U(\varepsilon F \circ F\eta) = U1_F = 1_{UF} \\ \mu \circ \eta T &= (U\varepsilon F) \circ (\eta UF) = (U\varepsilon \circ \eta U)F = 1_U F = 1_{UF} \end{aligned}$$

$$\begin{aligned} (U\varepsilon F) \circ (\eta UF) &= (U(\varepsilon(F(b))))(UF(\eta(b))) \\ &= U(\varepsilon(F(b))F(\eta(b))) = U(1_F) \end{aligned}$$

$$\begin{tikzcd} UFUFUF \arrow[r, "U\varepsilon FUF"] \arrow[d, "UFU\varepsilon F"'] & UFUF \arrow[d, "U\varepsilon F"] \\ UFUF \arrow[r, "U\varepsilon F"] & UF \end{tikzcd} \qquad \begin{tikzcd} FUFUF \arrow[r, "\varepsilon FUF"] \arrow[d, "FU\varepsilon F"'] & FUF \arrow[d, "\varepsilon F"] \\ FUF \arrow[r, "\varepsilon F"'] & F \end{tikzcd} \qquad \begin{tikzcd} FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"'] & a \arrow[d, "f"'] \\ FU(b) \arrow[r, "\varepsilon(b)"'] & b \end{tikzcd}$$

association law

$$\begin{aligned} \mu \circ \mu T &= \mu \circ T\mu \\ U\varepsilon(a)F \circ U\varepsilon(a)FFU &= U\varepsilon(a)F \circ FUU\varepsilon(a)F \end{aligned}$$

$U\varepsilon F \circ U\varepsilon FFU = U\varepsilon F \circ FUU\varepsilon F$
naturality of ε
$\varepsilon(b)FU(f)(a) = f\varepsilon(a)$

$a = FUF(a), b = F(a), f = \varepsilon F$

$$
\begin{aligned}
\varepsilon(F(a))(FU(\varepsilon F))(a) &= (\varepsilon F)(\varepsilon FUF(a)) \\
U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\
U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\
U\varepsilon F \circ FUU\varepsilon F &= U\varepsilon F \circ U\varepsilon FFU
\end{aligned}
$$

$$\begin{tikzcd}
FUF(a) \arrow[d, "\varepsilon(F(a))"] & FUFUF(a) \arrow[l, "FU(\varepsilon(F(a)))"'] \arrow[d, "\varepsilon(FUF(a))"] \\
F(a) & FUF(a) \arrow[l, "\varepsilon(F(a))"]
\end{tikzcd}$$

7 Eilenberg-Moore category

(T, η, μ)

A^T object (A, ϕ)

$\phi\eta(A) = 1_A, \phi\mu(A) = \phi T(\phi)$

arrow f.

$\phi T(f) = f\phi$

$$\begin{tikzcd}
a \arrow[r, "\eta(a)"] \arrow[dr, "1_a"] & T(a) \arrow[d, "\phi"] \\
& a
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(a) \arrow[r, "\mu(a)"] \arrow[d, "T(\phi)"'] & T(a) \arrow[d, "\phi"] \\
T(a) \arrow[r, "\phi"'] & T(a)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(a) \arrow[r, "T(f)"] \arrow[d, "\phi"'] & T(b) \arrow[d, "\phi'"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

8 EM on monoid

$f : a \to b$

$T : a \to (m, a)$

$\eta : a \to (1, a)$
$\mu : (m, (m', a)) \to (mm', a)$
$\phi : (m, a) \to \phi(m, a) = ma$

$$\begin{tikzcd}
a \arrow[r, "\eta(a)"] \arrow[dr, "1_a"] & (1, a) \arrow[d, "\phi"] \\
 & 1a
\end{tikzcd}$$

$$\begin{tikzcd}
(m, (m', a)) \arrow[r, "\mu(a)"] \arrow[d, "T(\phi)"'] & (mm', a) \arrow[d, "\phi"] \\
(m, m'a) \arrow[r, "\phi"'] & mm'a
\end{tikzcd}$$

$$\begin{tikzcd}
(m, a) \arrow[r, "T(f)"] \arrow[d, "\phi"'] & (m, f(a)) \arrow[d, "\phi'"] \\
ma \arrow[r, "f"'] & mf(a) = f(ma)
\end{tikzcd}$$

object (a, ϕ). arrow f.
$\phi T(f)(m, a) = f\phi(m, a)$
$\phi(m, f(a)) = f(a)$
$U^T : A^T \to A$
$U^T(a, \phi) = a, U^T(f) = f$
$F^T : A \to A^T$
$F^T(a) = (T(a), \mu(a)), F^T(f) = T(F)$

9 Comparison Functor K^T

$K^T(B) = (U(B), U\varepsilon(B))a, K^T(f) = U(g)$
$U^T K^T(b) = U(b)$
$U^T K^T(f) = U^T U(f) = U(f)$
$K^T F(a) = (UF(a), U\varepsilon(F(a))) = (T(a), \mu(a)) = F^T(a)$
$K^T F(f) = UF(f) = T(f) = F^T(f)$
$\eta U(a, \phi) = \eta(a), U\varepsilon(a, \phi) = \varepsilon^T K^T(b) = U\varepsilon(b)$

$$\begin{tikzcd}
B \arrow[r, "K^T"] \arrow[dr, leftrightarrow, "F", "U"'] & A^T \arrow[d, leftrightarrow, "U^T", "F^T"'] \\
A_T \arrow[u, "K_T"'] \arrow[r, leftrightarrow, "F_T", "U_T"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
& B \arrow[d, leftrightarrow, "F", "U"'] \arrow[dr, "K^T"] & \\
A_T \arrow[ur, "K_T"] \arrow[r, "U_T"] & A \arrow[r, "F^T"] & A^T
\end{tikzcd}$$

10 Kleisli Category

Object of A.

Arrow $f : a \to T(a)$ in A. In A_T, $f : b \to c, g : c \to d$,
$g * f = \mu(d)T(g)f$
$\eta(b) : b \to T(b)$ is an identity.
$f * \eta(b) = \mu(c)T(f)\eta(b) = \mu(c)\eta(T(c))f = 1_T(c)f = f$
and
$\eta(c) * f = \mu(c)T\eta(c)f = 1_T(c)f = f$
association law $g * (f * h) = (g * f) * h$,
$h : a \to T(b), f : b \to T(c), g : c \to T(d)$,
naturality of μ

$$\begin{tikzcd}
f * h & & & T(c) \arrow[r, leftarrow, "\mu(c)"] & T^2(c) \arrow[r, leftarrow, "T(f)"] & T(b) \arrow[r, leftarrow, "h"] & a \\
g * (f * h) & T(d) \arrow[r, leftarrow, "\mu(d)"] & T^2(d) \arrow[r, leftarrow, "T(g)"] & T(c) \arrow[r, leftarrow, "\mu(c)"] & T^2(c) \arrow[r, leftarrow, "T(f)"] & T(b) \arrow[r, leftarrow, "h"] & a \\
(g * f) * h & T(d) \arrow[r, leftarrow, "\mu(d)"] & T^2(d) \arrow[r, leftarrow, "\mu(d)T"] & T^3(d) \arrow[r, leftarrow, "T^2(g)"] & T^2(c) \arrow[r, leftarrow, "T(f)"] & T(b) \arrow[r, leftarrow, "h"] & a \\
(g * f) * h & T(d) \arrow[r, leftarrow, "\mu(d)"] & T^2(d) \arrow[r, leftarrow, "T\mu(d)"] & T^3(d) \arrow[r, leftarrow, "T^2(g)"] & T^2(c) \arrow[r, leftarrow, "T(f)"] & T(b) \arrow[r, leftarrow, "h"] & a \\
g * f & T(d) \arrow[r, leftarrow, "\mu(d)"] & T^2(d) \arrow[r, leftarrow, "T(g)"] & T(c) \arrow[r, leftarrow, "f"] & b
\end{tikzcd}$$

$$\begin{tikzcd}
T^2(d) \arrow[d, "\mu(d)"'] \arrow[r, leftarrow, "T\mu(d)"] & T^2(T(d)) \arrow[d, "\mu(T(d))"] \arrow[r, leftarrow, "T^2(g)"] & T^2(c) \arrow[d, "\mu(c)"] \\
T(d) \arrow[r, leftarrow, "\mu(d)"] & T(T(d)) \arrow[r, leftarrow, "T(g)"] & T(c)
\end{tikzcd}$$

$$\begin{aligned}
g * (f * h) &= g * (\mu(c)T(f)h) \\
&= \mu(d)(T(g))(\mu(c)T(f)h) \\
&= \mu(d)T(g)\mu(c)T(f)h
\end{aligned}$$

$$\begin{aligned}
(g * f) * h &= (\mu(d)T(g)f) * h \\
&= \mu(d)T(\mu(d)T(g)f)h \\
&= \mu(d)T(\mu(d))T^2(g)T(f)h
\end{aligned}$$

$\mu(d)\mu(d)T = \mu(d)T\mu(d)$
$\mu(T(d))T^2(g) = T(g)\mu(c)$ naturality of μ.
$\mu(d)T\mu(d)T^2(g) = \mu(d)\mu(T(d))T^2(g) = \mu(d)T(g)\mu(c)$

$$\begin{tikzcd}
T^3(d) \arrow[d, "\mu(T(d))"] & T^2(c) \arrow[l, "T^2(g)"'] \\
T^2(d) & T(c) \arrow[l, "T(g)"] \arrow[u, "\mu(c)"]
\end{tikzcd}$$

$\mu(T(d)) = T\mu(d)?$
$(m, (m'm'', a)) = (mm', (m'', a))$ No, but
$\mu\mu(T(d)) = \mu T\mu(d).$

11 Ok

$T(g)\mu(c) = T(\mu(d))T^2(g)$ であれば良いが。
$\mu(d)T^2(g) = T(g)\mu(c)$
ちょっと違う。 $\mu(d)T(\mu(d))T^2(g)$ が、
$\mu(d)\mu(d)T^2(g)$
となると良いが。
$\mu(d)T(\mu(d)) = \mu(d)\mu(T(d))$

12 monoid in Kleisli category

$T : a \to (m, a)$
$T : f \to ((m, a) \to (m, f(a)))$
$\mu(a) : (m, (m', a)) \to (mm', a)$
$f : a \to (m, f(a))$

$$\begin{aligned}
g * f(b) &= \mu(d)T(g)f(b) = \mu(d)T(g)(m, f(b)) \\
&= \mu(m, (m', gf(b))) = (mm', gf(b)) \\
(g * f) * h(a) &= \mu(d)T(\mu(d)T(g)f)h(a) = \mu(d)T(\mu(d))(TT(g))T(f)(m, h(a)) \\
&= \mu(d)T(\mu(d))(TT(g))(m, (m', fh(a))) \\
&= \mu(d)T(\mu(d)(m, (m', (m'', gfh(a))))) = (mm'm'', gfh(a)) \\
g * (f * h)(a) &= (\mu(d)(T(g)))\mu(c)T(f)h(a) = (\mu(d)(T(g)))\mu(c)T(f)(m, h(a)) \\
&= (\mu(d)(T(g)))\mu(c)(m, (m', fh(a)))
\end{aligned}$$

$$= \quad \mu(d)T(g)(mm', fh(a)) = (mm'm'', gfh(a))$$

13 Resolution of Kleiseli category

$f : a \to b, g : b \to c$
$U_T : A_T \to A$
$U_T(a) = T(a)$
$U_T(f) = \mu(b)T(f)$
$g * f = \mu(d)T(g)f$

$$\begin{aligned}
U_T(g * f) &= U_T(\mu(c)T(g)f) \\
&= \mu(c)T(\mu(c)T(g)f) \\
&= \mu(c)\mu(c)T(T(g)f)) = \mu(c)\mu(c)TT(g)T(f) \text{ association law} \\
U_T(g)U_T(f) &= \mu(c)T(g)\mu(b)T(f) = \mu(c)\mu(c)TT(g)T(f) \\
T(g)\mu(b) &= \mu(c)TT(g)
\end{aligned}$$

$$\begin{tikzcd}
TT \arrow[d, "\mu(c)"] & TT \arrow[l, "TT(g)"'] \arrow[d, "\mu(b)"] \\
T & T \arrow[l, "T(g)"']
\end{tikzcd}$$

$F_T : A \to A_T$
$F_T(a) = a$
$F_T(f) = \eta(b)f$
$F_T(1_a) = \eta(a) = 1_{F_T(a)}$

$$\begin{aligned}
F_T(g) * F_T(f) &= \mu(c)T(F_T(g))F_T(f) \\
&= \mu(c)T(\eta(c)g)\eta(b)f \\
&= \mu(c)T(\eta(c))T(g)\eta(b)f \\
&= T(g)\eta(b)f \text{ unity law} \\
&= \eta(c)gf = F_T(gf)
\end{aligned}$$

$\eta(c)g = T(g)\eta(b)$

$$\begin{tikzcd}
c \arrow[d, "\eta(c)"] & b \arrow[l, "g"'] \arrow[d, "\eta(b)"] \\
T & T \arrow[l, "T(g)"']
\end{tikzcd}$$

$\mu \circ T\eta = 1_T = \mu \circ \eta T$ Unity law

$$\varepsilon_T(a) \quad = \quad 1_{T(a)}$$

$$U_T\varepsilon_T F_T \quad = \quad \mu$$

$$U_T\varepsilon_T F_T a(a) \quad = \quad U_T\varepsilon_T(a) = U_T(1_{T(a)} = \mu(a)$$

$$\begin{aligned}
\varepsilon_T(F_T(a)) * F_T(\eta(a)) &= \varepsilon_T(a) * F_T(\eta(a)) \\
&= 1_{T(a)} * (F_T(\eta(a))) \\
&= 1_{T(a)} * (\eta(T(a))\eta(a))) \\
&= \mu(T(a))T(1_{T(a)})(\eta(T(a))\eta(a))) \\
&= \mu(T(a))\eta(T(a))\eta(a) \\
&= \eta(a) = 1_{F_T}
\end{aligned}$$

$$\begin{aligned}
U_T(\varepsilon_T(a))\eta(U_T(a)) &= U_T(1_{T(a)}\eta(T(a))) \\
&= \mu(T(a))T(1_{T(a)})\eta(T(a)) \\
&= \mu(T(a))\eta(T(a))1_{T(a)} \\
&= 1_{T(a)} = T(1_a) = 1_{U_T}
\end{aligned}$$

13.1 Comparison functor K_T

Adjoint (B, U, F, ε), $K_T : A_T \to B$,
$g : b \to c$.

$$\begin{aligned}
K_T(a) &= F(a) \\
K_T(g) &= \varepsilon(F(c))F(g) \\
K_T F_T(a) &= K_T(a) = F(a) \\
K_T F_T(f) &= K_T(\eta(b)f) \\
&= \varepsilon(F(b))F(\mu(b)f) \\
&= \varepsilon(F(b))F(\mu(b))F(f) = F(f)
\end{aligned}$$

$$\begin{aligned}
K_T(\eta(b)) &= \varepsilon(F(b))F(\eta(b)) = 1_{F(b)} \\
K_T(\eta(T(c))g) &= \varepsilon(F(T(c)))F(\eta(T(c))g) = F(g) \\
K_T(g)K_T(f) &= \varepsilon(F(c))F(g)\varepsilon(F(b))F(f) = \varepsilon(F(c))\varepsilon(F(c))FUF(g)F(f) \\
K_T(g * f) &= \varepsilon(F(c))F(\mu(c)UF(g)f) = \varepsilon(F(c))F(\mu(c))FUF(g)F(f) \\
\varepsilon(F(c))FUF(g) &= F(g)\varepsilon(F(b))
\end{aligned}$$

$$\begin{tikzcd}
FU(F(c)) \arrow[d, "\varepsilon(F(c))"] & FU(F(b)) \arrow[l, "FU(F(g))"'] \arrow[d, "\varepsilon(F(b))"] \\
F(c) & F(b) \arrow[l, "F(g)"']
\end{tikzcd}$$

$\varepsilon(F(c))F(\mu(c)) = \varepsilon(F(c))\varepsilon(F(c))$?
$\varepsilon(F(c))F(\mu(c)) = \varepsilon(F(c))FU\varepsilon(F(c))$

$$\begin{tikzcd}
FUFU(c) \arrow[d, "\varepsilon F(c))"] & FUFU(F(c)) \arrow[l, "FU\varepsilon(F(c))"'] \arrow[d, "\varepsilon(F(c))"] \\
FU(c) & FU(F(c)) \arrow[l, "\varepsilon(F(c))"']
\end{tikzcd}$$

$$UK_T(a) = UF(a) = T(a) = U_T(a)$$
$$UK_T(g) = U(\varepsilon((F(c))F(g))) = U(\varepsilon(F(c)))UF(g) = \mu(c)T(g) = U_T(g)$$

14 Monoid

$T : A \to MxA$

$T(a) = (m, a)$

$T(f) : T(A) \to T(f(A))$

$T(f)(m, a) = (m, f(a))$

$T(fg)(m, a) = (m, fg(a))$

15 association of Functor

$T(f)T(g)(m, a) = T(f)(m, g(a)) = (m, fg(a)) = T(fg)(m, a)$

$\mu : TxT \to T$

$\mu_a(T(T(a)) = \mu_A((m, (m', a))) = (m * m', a)$

16 TT

$TT(a) = (m, (m', a))$

$TT(f)(m, (m', a)) = (m, (m', f(a))$

17 naturality of μ

$$\begin{tikzcd}
TT(a) \arrow[r, "\mu(a)"] \arrow[d, "TT(f)"] & T(a) \arrow[d, "T(f)"] \\
TT(b) \arrow[r, "\mu(b)"] & T(b)
\end{tikzcd}$$

$\mu(b)TT(f)TT(a) = T(f)\mu(a)TT(a)$
$\mu(b)TT(f)TT(a) = \mu(b)((m,(m',f(a))) = (m * m', f(a))$
$T(f)\mu(a)(TT(a)) = T(f)(m * m', a) = (m * m', f(a))$

18 $\boldsymbol{\mu} \bigcirc \boldsymbol{\mu}$

Horizontal composition of μ
$f \rightarrow \mu_T T(a)$
$a \rightarrow TT(a)$
$\mu_T(a)TTT(a) = \mu_T(a)(m,(m',(m'',a))) = (m * m',(m'',a))$

$$\begin{tikzcd}
TTTT(a) \arrow[r, "\mu(TTT(a))"] \arrow[d, "TT(\mu)"'] & TTT(a) \arrow[d, "T(\mu)"] \\
TTT(a) \arrow[r, "\mu(TT(a))"] & TT(a)
\end{tikzcd}$$

$$\begin{aligned}
T(\mu_a)\mu_a TTTT(a) &= T(\mu_a)\mu_a(m_0,(m_1,(m_2,(m_3,a))))) \\
&= T(\mu_a)(m_0 * m_1,(m_2,(m_3,a))) = (m_0 * m_1,(m_2 * m_3,a)) \\
\mu_b TT(\mu_a)TTTT(a) &= \mu_b TT(\mu_a)(m_0,(m_1,(m_2,(m_3,a))))) \\
&= \mu_b(m_0,(m_1,(m_2 * m_3,a))) = (m_0 * m_1,(m_2 * m_3,a))
\end{aligned}$$

Horizontal composition of natural transformation

19 Natural transformation ε and Functor $F : A \to B, U : B \to A$

$\varepsilon : FUFU \to FU$

$\varepsilon : FU \to 1_B$

Naturality of ε

$$\begin{tikzcd}
FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$$\varepsilon(b)FU(f)a = f\varepsilon(a)a$$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[d, "FU(f)"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b)
\end{tikzcd}$$

$$\varepsilon((FU(b))FUFU(f)FU(a) = FU(f)\varepsilon(FU(a))FU(a)$$

20 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

21 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

22 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

23 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

24 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

25 Vertial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{tikzcd}
FUFU(a) \arrow[r, "\varepsilon(FU(a))"] \arrow[d, "FUFU(f)"] & FU(a) \arrow[r, "\varepsilon(a)"] \arrow[d, "FU(f)"] & a \arrow[d, "f"] \\
FUFU(b) \arrow[r, "\varepsilon(FU(b))"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

26 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd}
FUFU & & FU \arrow[ll] & & B \arrow[ll] \\
& FU \arrow[d, "\varepsilon"] & & FU \arrow[d, "\varepsilon"] & \\
& 1_B & & 1_B & \\
B & & B \arrow[ll] & & B \arrow[ll]
\end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\
FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B
\end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

27 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd}
FUFU & & FU \arrow[ll] & & B \arrow[ll] \\
& FU \arrow[d, "\varepsilon"] & & FU \arrow[d, "\varepsilon"] & \\
& 1_B & & 1_B & \\
B & & B \arrow[ll] & & B \arrow[ll]
\end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\
FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B
\end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$

$$\begin{tikzcd}
b \arrow[r, "U"] & U(b) \arrow[r, "F"] & FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

28 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd}
FUFU & & FU \arrow[ll] & & B \arrow[ll] \\
& FU \arrow[d, "\varepsilon"] & & FU \arrow[d, "\varepsilon"] & \\
& 1_B & & 1_B & \\
B & & B \arrow[ll] & & B \arrow[ll]
\end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\
FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B
\end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

29 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd}
FUFU & & FU \arrow[ll] & & B \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}
FU \arrow[d, "\varepsilon"] & & FU \arrow[d, "\varepsilon"] \\
1_B & & 1_B
\end{tikzcd}$$

$$\begin{tikzcd}
B & & B \arrow[ll] & & B \arrow[ll]
\end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\
FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B
\end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$

$$\begin{tikzcd} b \arrow[r, "U"] & U(b) \arrow[r, "F"] & FU(b) \arrow[r, "\varepsilon(b)"] & b \end{tikzcd}$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

30 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd} FUFU & & FU \arrow[ll] & & B \arrow[ll] \\ & FU \arrow[d, "\varepsilon"] & & FU \arrow[d, "\varepsilon"] & \\ & 1_B & & 1_B & \\ B & & B \arrow[ll] & & B \arrow[ll] \end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd} FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\ FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B \end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$
$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

31 Horizontal Composition $\varepsilon \circ \varepsilon$

$$\begin{tikzcd}
FUFU & FU \arrow[l] & B \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
FU \arrow[d, "\varepsilon"] & FU \arrow[d, "\varepsilon"] \\
1_B & 1_B
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \arrow[l] & B \arrow[l]
\end{tikzcd}$$

cf. $FUFU, FU$ has objects of B.
$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & 1_A \arrow[d, "1_a\varepsilon(b)"] \\
FU1_B(b) \arrow[r, "\varepsilon(b)"] & 1_B
\end{tikzcd}$$

that is

$$\begin{tikzcd}
FUFU(b) \arrow[r, "\varepsilon FU(b)"] \arrow[d, "FU\varepsilon(b)"] & FU(b) \arrow[d, "\varepsilon(b)"] \\
FU(b) \arrow[r, "\varepsilon(b)"] & b
\end{tikzcd}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B
$\varepsilon : FU \to 1_B$
$\varepsilon(b) : FU(b) \to b$

$$\begin{tikzcd} b \arrow[r, "U"] & U(b) \arrow[r, "F"] & FU(b) \arrow[r, "\varepsilon(b)"] & b \end{tikzcd}$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

32 Yoneda Functor

$Y : A \rightarrow Sets^{A^{op}}$
$Hom_A : A^{op} \times A \rightarrow Sets$
$g : a' \rightarrow a, h : b \rightarrow b'$
$Hom_A((g,h)) : Home_A(a,b) \rightarrow \{hfg | f \in Home_A(a,b)\}$
$Hom_A((g,h) \circ (g',h') : Home_A(a,b) \rightarrow \{hh'fgg' | f \in Home_A(a,b)\}$
$Hom_A((g,h))Hom_A((g',h')) : Home_A(a,b) \rightarrow \{h'fg' | f \in Home_A(a,b)\} \rightarrow \{hh'fgg' | f \in Home_A(a,b)\}$

$$\begin{tikzcd} a \arrow[r, "g'"] & a' \arrow[r, "g"] \arrow[d] & a'' \arrow[d, "f"] \\ b & b' \arrow[l, "h'"'] & b'' \arrow[l, "h"'] \end{tikzcd}$$

$Hom^*_A : A^{op} \rightarrow Sets^A$
$f^{op} : a \rightarrow c(f : c \rightarrow a)$
$g^{op} : c \rightarrow d(g : d \rightarrow c)$
$Home^*_A(a) : a \rightarrow \lambda b.Hom_A(a,b)$
$Home^*_A(f^{op}) : (a \rightarrow \lambda b.Hom_A(a,b)) \rightarrow (c \rightarrow \lambda b.Hom_A(f(c),b))$
$Home^*_A(g^{op}f^{op}) : (a \rightarrow \lambda b.Hom_A(a,b)) \rightarrow (d \rightarrow \lambda b.Hom_A(fg(d),b))$
$Home^*_A(g^{op})Home^*_A(f^{op}) : (a \rightarrow \lambda b.Hom_A(a,b)) \rightarrow (c \rightarrow \lambda b.Hom_A(f(c),b)) \rightarrow (d \rightarrow \lambda b.Hom_A(fg(d),b))$

$Hom^*_{A^{op}} : A \to Sets^{A^{op}}$

$f : c \to b \; g : d \to c$

$Home^*_{A^{op}}(b) : b \to \lambda a.Hom_{A^{op}}(a, b)$

$Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c)))$

$Home^*_{A^{op}}(gf) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (d \to \lambda a.Hom_{A^{op}}(a, gf(d)))$

$Home^*_{A^{op}}(g)Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c))) \to (d \to \lambda a.Hom_{A^{op}}(a, gf(d)))$

Arrows in $Set^{A^{op}}$?

$f : b \to c = (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c)))$

$Set^{A^{op}} : A^{op} \to Set$

an object $b = \lambda a.Hom_{A^{op}}(a, b)$ is a functor from A^{op} to Set.

$t : (\lambda a.Hom_{A^{op}}(a, b)) \to (\lambda a.Hom_{A^{op}}(a, t(c)))$ should be a natural transformatin.

$f^{op} : (b : A^{op}) \to (c : A^{op}) = f : c \to b$

$$\begin{tikzcd}
Hom_{A^{op}}(a, c) \arrow[r, "t(c)"] \arrow[d, "Home^* A^{op}(a{,}f)"] & Hom_{A^{op}}(a, t(c)) \\
Hom_{A^{op}}(a, b) \arrow[r, "t(b)"] & Hom_{A^{op}}(a, t(b)) \arrow[u, "Home^* A^{op}(a{,}f)"']
\end{tikzcd}$$

32.1 Contravariant functor

$h_a = Hom_A(-, a)$

$f : b \to c, Hom_A(f, 1_a) : Hom_A(c, a) \to Hom_A(b, a)$

33 Yoneda Functor

$Y : A \to Sets^{A^{op}}$

$Hom_A : A^{op} \times A \to Sets$

$g : a' \to a, h : b \to b'$

$Hom_A((g, h)) : Home_A(a, b) \to \{hfg | f \in Home_A(a, b)\}$

$Hom_A((g, h) \circ (g', h') : Home_A(a, b) \to \{hh'fgg' | f \in Home_A(a, b)\}$

$Hom_A((g, h))Hom_A((g', h')) : Home_A(a, b) \to \{h'fg' | f \in Home_A(a, b)\} \to \{hh'fgg' | f \in Home_A(a, b)\}$

$$\begin{tikzcd}
a \arrow[r, "g'"] & a' \arrow[r, "g"] \arrow[d] & a'' \arrow[d, "f"] \\
b & b' \arrow[l, "h'"'] & b'' \arrow[l, "h"']
\end{tikzcd}$$

$Hom^*_A : A^{op} \rightarrow Sets^A$

$f^{op} : a \rightarrow c(f : c \rightarrow a)$

$g^{op} : c \rightarrow d(g : d \rightarrow c)$

$Home^*_A(a) : a \rightarrow \lambda b.Hom_A(a, b)$

$Home^*_A(f^{op}) : (a \rightarrow \lambda b.Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b.Hom_A(f(c), b))$

$Home^*_A(g^{op} f^{op}) : (a \rightarrow \lambda b.Hom_A(a, b)) \rightarrow (d \rightarrow \lambda b.Hom_A(fg(d), b))$

$Home^*_A(g^{op})Home^*_A(f^{op}) : (a \rightarrow \lambda b.Hom_A(a, b)) \rightarrow (c \rightarrow \lambda b.Hom_A(f(c), b)) \rightarrow (d \rightarrow \lambda b.Hom_A(fg(d), b))$

$Hom^*_{A^{op}} : A \rightarrow Sets^{A^{op}}$

$f : c \rightarrow b \ g : d \rightarrow c$

$Home^*_{A^{op}}(b) : b \rightarrow \lambda a.Hom_{A^{op}}(a, b)$

$Home^*_{A^{op}}(f) : (b \rightarrow \lambda a.Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a.Hom_{A^{op}}(a, f(c)))$

$Home^*_{A^{op}}(gf) : (b \rightarrow \lambda a.Hom_{A^{op}}(a, b)) \rightarrow (d \rightarrow \lambda a.Hom_{A^{op}}(a, gf(d)))$

$Home^*_{A^{op}}(g)Home^*_{A^{op}}(f) : (b \rightarrow \lambda a.Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a.Hom_{A^{op}}(a, f(c))) \rightarrow (d \rightarrow \lambda a.Hom_{A^{op}}(a, gf(d)))$

Arrows in $Set^{A^{op}}$?

$f : b \rightarrow c = (b \rightarrow \lambda a.Hom_{A^{op}}(a, b)) \rightarrow (c \rightarrow \lambda a.Hom_{A^{op}}(a, f(c)))$

$Set^{A^{op}} : A^{op} \rightarrow Set$

an object $b = \lambda a.Hom_{A^{op}}(a, b)$ is a functor from A^{op} to Set.

$t : (\lambda a.Hom_{A^{op}}(a, b)) \rightarrow (\lambda a.Hom_{A^{op}}(a, t(c)))$ should be a natural transformatin.

$f^{op} : (b : A^{op}) \rightarrow (c : A^{op}) = f : c \rightarrow b$

$$\begin{tikzcd}
Hom_{A^{op}}(a, c) \arrow[r, "t(c)"] \arrow[d, "Home^*A^{op}(a{,}f)"] & Hom_{A^{op}}(a, t(c)) \\
Hom_{A^{op}}(a, b) \arrow[r, "t(b)"] & Hom_{A^{op}}(a, t(b)) \arrow[u, "Home^*A^{op}(a{,}f)"]
\end{tikzcd}$$

33.1 Contravariant functor

$h_a = Hom_A(-, a)$

$f : b \to c, Hom_A(f, 1_a) : Hom_A(c, a) \to Hom_A(b, a)$

34 Yoneda Functor

$Y : A \to Sets^{A^{op}}$

$Hom_A : A^{op} \times A \to Sets$

$g : a' \to a, h : b \to b'$

$Hom_A((g, h)) : Home_A(a, b) \to \{hfg | f \in Home_A(a, b)\}$

$Hom_A((g, h) \circ (g', h') : Home_A(a, b) \to \{hh'fgg' | f \in Home_A(a, b)\}$

$Hom_A((g, h))Hom_A((g', h')) : Home_A(a, b) \to \{h'fg' | f \in Home_A(a, b)\} \to \{hh'fgg' | f \in Home_A(a, b)\}$

$$\begin{tikzcd}
a \arrow[r, "g'"] & a' \arrow[r, "g"] \arrow[d] & a'' \arrow[d, "f"] \\
b & b' \arrow[l, "h'"'] & b'' \arrow[l, "h"']
\end{tikzcd}$$

$Hom^*_A : A^{op} \to Sets^A$

$f^{op} : a \to c(f : c \to a)$

$g^{op} : c \to d(g : d \to c)$

$Home^*_A(a) : a \to \lambda b.Hom_A(a, b)$

$Home^*_A(f^{op}) : (a \to \lambda b.Hom_A(a, b)) \to (c \to \lambda b.Hom_A(f(c), b))$

$Home^*_A(g^{op} f^{op}) : (a \to \lambda b.Hom_A(a, b)) \to (d \to \lambda b.Hom_A(fg(d), b))$

$Home^*_A(g^{op})Home^*_A(f^{op}) : (a \to \lambda b.Hom_A(a, b)) \to (c \to \lambda b.Hom_A(f(c), b)) \to (d \to \lambda b.Hom_A(fg(d), b))$

$Hom^*_{A^{op}} : A \to Sets^{A^{op}}$

$f : c \to b\ g : d \to c$

$Home^*_{A^{op}}(b) : b \to \lambda a.Hom_{A^{op}}(a, b)$

$Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c)))$

$Home^*_{A^{op}}(gf) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (d \to \lambda a.Hom_{A^{op}}(a, gf(d)))$

$Home^*_{A^{op}}(g)Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c))) \to (d \to \lambda a.Hom_{A^{op}}(a, gf(d)))$

Arrows in $Set^{A^{op}}$?

$f : b \to c = (b \to \lambda a.Hom_{A^{op}}(a, b)) \to (c \to \lambda a.Hom_{A^{op}}(a, f(c)))$

$Set^{A^{op}} : A^{op} \to Set$

an object $b = \lambda a.Hom_{A^{op}}(a, b)$ is a functor from A^{op} to Set.

$t : (\lambda a.Hom_{A^{op}}(a, b)) \to (\lambda a.Hom_{A^{op}}(a, t(c)))$ should be a natural transformatin.

$f^{op} : (b : A^{op}) \to (c : A^{op}) = f : c \to b$

$$\begin{tikzcd}
Hom_{A^{op}}(a, c) \arrow[r, "t(c)"] \arrow[d, "Home^{*} A^{op}(a,f)"] & Hom_{A^{op}}(a, t(c)) \\
Hom_{A^{op}}(a, b) \arrow[r, "t(b)"] & Hom_{A^{op}}(a, t(b)) \arrow[u, "Home^{*} A^{op}(a,f)"']
\end{tikzcd}$$

34.1 Contravariant functor

$h_a = Hom_A(-, a)$

$f : b \to c, Hom_A(f, 1_a) : Hom_A(c, a) \to Hom_A(b, a)$